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На английском языке

I. SUVOROV

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A. BASIC ANALYTIC GEOMETRY IN THE PLANE

CHAPTER I

METHOD OF COORDINATES

Sec. 1. The Coordinates of a Point

1°. Analytic geometry in the plane treats of straight lines, circles, ellipses, hyperbolas and parabolas. These are studied by the method of coordinates.

The *method of coordinates* specifies with numbers the position of a point relative to coordinate axes.

The *coordinate axis* is a straight line (Ox in Fig. 1) where we have:

1) point O, the origin,
2) a positive direction (from left to right in Fig. 1),
3) a unit for measuring lengths, l, which is also called a scale unit.

The distance of any point M on the axis from the origin O (the segment OM) can be measured by the unit l, i.e., it can be

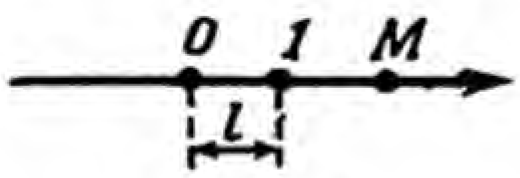

Fig. 1.

expressed by a certain number. The plus sign is assigned to this number if the direction from the origin O to the point M coincides with the positive direction of the axis, and the minus sign is affixed if it is opposite to the positive direction of the axis. Thus, to every point M of the coordinate axis there corresponds some definite number. It is called the coordinate of the point M. We note that the coordinate of the origin O is equal to zero.

The converse is also true: to every real number there corresponds a point on the coordinate axis whose coordinate is expressed by this number for the given unit of measurement.

2°. Two mutually perpendicular coordinate axes Ox and Oy, intersecting at O (Fig. 2) form a rectangular or Cartesian system of coordinates *.

The positive direction of each axis is indicated in the figure by an arrow.

The axes Ox and Oy divide the plane into four parts called quadrants, which are numbered as follows: the first quadrant is that part of the plane between $+Ox$ and $+Oy$; the second quadrant, between $+Oy$ and $-Ox$; the third quadrant, between $-Ox$ and $-Oy$, and the fourth quadrant, between $-Oy$ and $+Ox$.

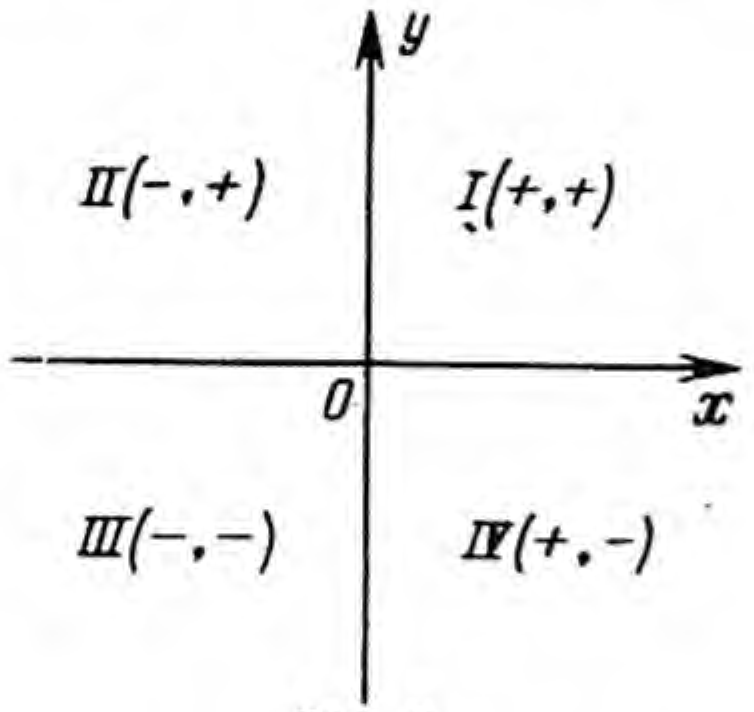

Fig. 2.

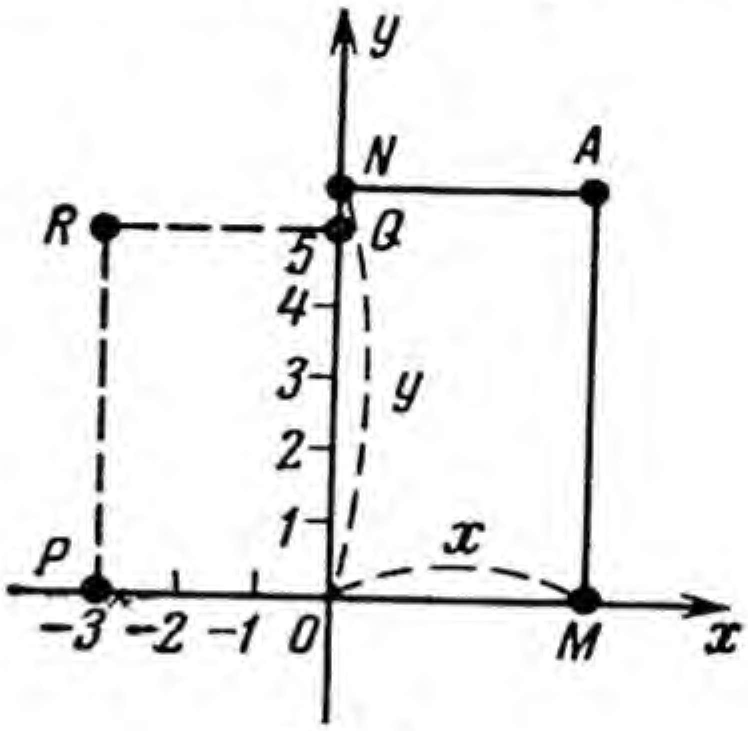

Fig. 3.

In order to specify with the help of numbers the location of a point A in the plane with respect to a given coordinate system (Fig. 3), we drop a perpendicular AM from the point A on the axis Ox and a perpendicular AN on the axis Oy. The points M and N, which are the bases of the perpendiculars dropped from the point A on the axes Ox and Oy respectively, are called the projections of A on the coordinate axis. Let x be the coordinate of the point M on the axis Ox and let y be the coordinate of the point N on the axis Oy.

Definition. *The number x, which is the coordinate of the projection of the point A on the axis Ox, is called the abscissa of the point A. The number y, which is the coordinate of the projection of the point A on the axis Oy, is called the ordinate of the point A***.

Accordingly, the axis Ox is called the axis of abscissas and the axis Oy, the axis of ordinates. For the sake of brevity, the phrase

---

* The rectangular coordinate system is called Cartesian in honour of the French mathematician René Descartes (Cartesius) who was first to publish (in 1637) a treatise on analytic geometry where he expressed the fundamental idea of the subject, namely, that the equation between x and y defines a line.

** The Latin word "abscissa" means "cut off", "ordinate"—"ordered".

"the point A has coordinates x and y" is usually written "the point $A(x, y)$" first the abscissa and then the ordinate. The signs of the abscissa and the ordinate of a point A are determined by the quadrant in which it lies. Fig. 2 shows the signs of the abscissa and the ordinate for each quadrant; the first sign is that of the abscissa, the second, that of the ordinate.

If the point A lies on the axis Ox (Oy), its ordinate (or abscissa) is zero.

3°. In this way we can determine the coordinates of any point in the plane. Conversely, the coordinates being given, we can specify the position of any point in the plane. For example, let $(-3, 5)$ represent the coordinates of a point R. Laying off 3 scale units to the left of the origin O along the axis Ox and 5 scale units upwards from the origin along the axis Oy, we obtain a point P on Ox and a point Q on Oy (Fig. 3); drawing the lines PR and QR, parallel to the axes Oy and Ox, respectively, we get the desired point $R(-3, 5)$ at the point of intersection.

4°. The coordinates x and y of the point A in the plane are numbers representing, respectively, the ratio of the segments OM and ON to the scale unit. Taking l as the scale unit for both axes, we have

$$x = \frac{OM}{l}, \quad y = \frac{ON}{l}. \tag{1}$$

These relations are usually written, in abbreviated form,

$$x = OM, \quad y = ON,$$

it being understood that in this case OM and ON do not represent the directed segments OM and ON but the numbers expressing their lengths when $l = 1$. This notation is convenient not only in that it is simpler but also because the coordinates x and y are represented pictorially.

Sec. 2. The Sum of Two Directed Segments

In analytic geometry, segments are characterised not only by their length but also by their direction. If with respect to two segments: 1) the straight lines representing the segments are parallel or coincident, 2) the direction from the origin (beginning) of the segment to its terminus (end) is the same (or opposite), the two segments are said to be identically (or oppositely) directed.

In designating directed segments by two letters, the first letter denotes the origin and the second the terminus of the segment. Two segments AB and BA are thus of the same length but of opposite direction:

$$AB = -BA.$$

The direction of a segment is sometimes indicated by an arrowhead attached to the terminus of the segment (Fig. 4*a* and *b*).

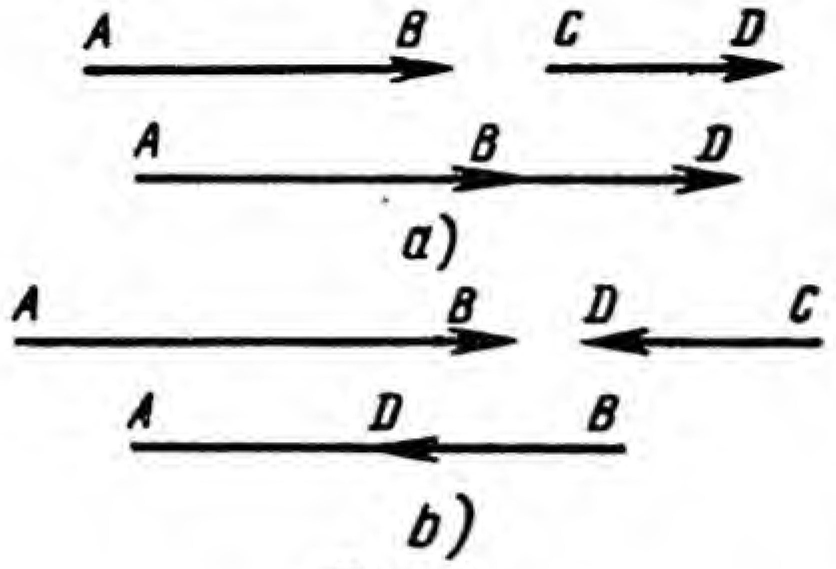

Fig. 4.

To obtain the sum of two identically or oppositely directed segments AB and CD (Fig. 4*a* and *b*), it is necessary to place them on a single straight line so that the origin of the second segment (point C) coincides with the terminus of the first segment (point B). Then the origin of the first segment (point A) and the terminus of the second segment (point D) will represent, respectively, the origin and the terminus of the segment, which is the sum of AB and CD:

$$AB + CD = AD.$$

Sec. 3. The Distance Between Two Points

1°. **Theorem.** *Whatever the position, on an axis, of two given points M_1 and M_2 relative to each other and to the coordinate origin, the magnitude of the segment M_1M_2 is given by the difference between the coordinates of the terminus and origin of the segment; in other words, if the origin and terminus of the segment M_1M_2 have coordinates x_1 and x_2 (or y_1 and y_2) then*

$$\boxed{M_1M_2 = x_2 - x_1 \quad \text{or} \quad M_1M_2 = y_2 - y_1} \tag{I}$$

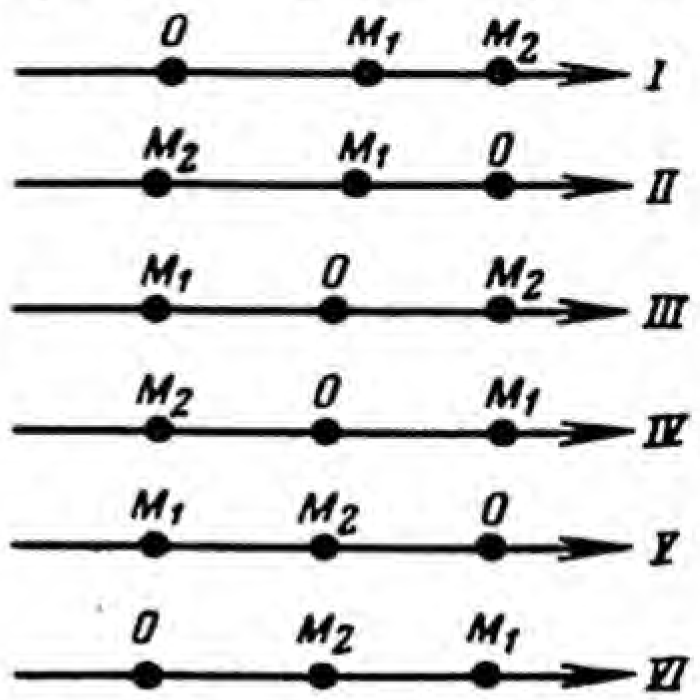

Fig. 5.

Proof. There are six different possible cases of the location of points M_1 and M_2 with respect to each other and the origin O (Fig. 5).

If reckoning from left to right, the points are in the order indicated on axes I and II, i.e., O, M_1, M_2 or M_2, M_1, O, then in both cases $OM_1 + M_1M_2 = OM_2$.

Whence, $M_1M_2 = OM_2 - OM_1$.

But $OM_1 = x_1$ and $OM_2 = x_2$. Therefore, in both cases, $M_1M_2 = x_2 - x_1$.

If reckoning from left to right, the points have the order indicated on axes III and IV, i. e.,

$$M_1,\ O,\ M_2 \text{ or } M_2,\ O,\ M_1,$$

then $M_1M_2 = M_1O + OM_2$.

But $M_1O = -OM_1 = -x_1$, $OM_2 = x_2$.

Therefore, $M_1M_2 = x_2 - x_1$.

When the points are arranged in the order indicated on axes V and VI, i. e., M_1, M_2, O or O, M_2, M_1, we have

$$M_1M_2 + M_2O = M_1O.$$

Whence $M_1M_2 = M_1O - M_2O$, or $M_1M_2 = OM_2 - OM_1$, because $M_1O = -OM_1$ and $M_2O = -OM_2$. But $OM_2 = x_2$ and $OM_1 = x_1$. Therefore, $M_1M_2 = x_2 - x_1$.

2°. Examples. 1. The magnitude of a segment whose origin and terminus are the points $M_1(3, 0)$ and $M_2(5, 0)$ is

$$M_1M_2 = 5 - 3 = 2.$$

2. If the points are $M_1(-3, 0)$ and $M_2(-5, 0)$, then $M_1M_2 = -5-(-3) = -5+3 = -2$.

3. If the points are $M_1(0, -2)$ and $M_2(0, 3)$, then $M_1M_2 = 3-(-2) = 3+2 = 5$.

3°. Corollary. *The magnitude of segment AB* (Fig. 6) *parallel to the axis of abscissas (or ordinates) is equal to the difference*

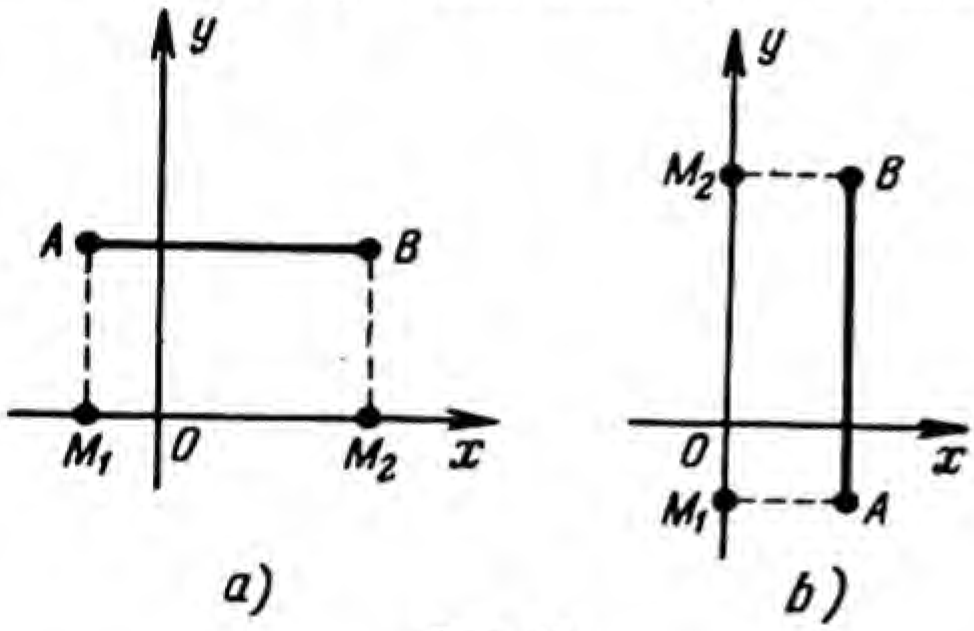

Fig. 6.

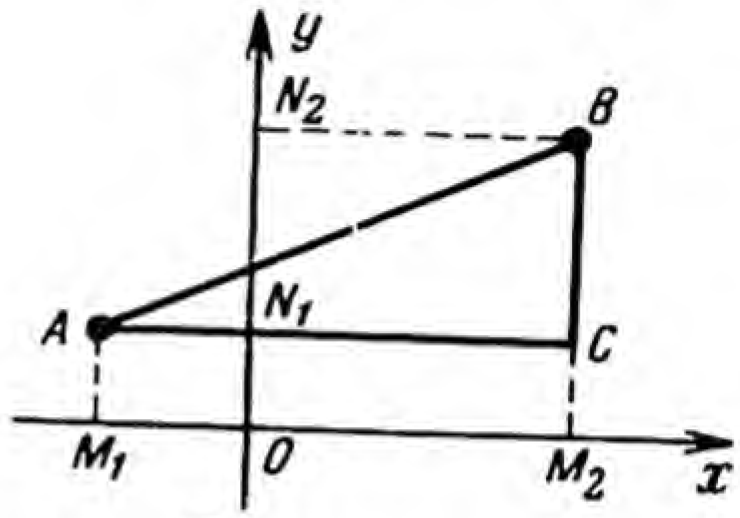

Fig. 7.

between the abscissas (or ordinates) of the terminus and origin of the segment, i.e.,

$$\boxed{AB = x_2 - x_1 \quad \text{or} \quad AB = y_2 - y_1} \qquad \text{(I)}$$

Indeed, dropping two perpendiculars AM_1 and BM_2 from these points onto the axis of abscissas (Fig. 6*a*) or onto the axis of ordinates (Fig. 6*b*) we see that the segment AB is equal in length to the segment M_1M_2 (as opposite sides of a rectangle) and is in the same direction. That is,

$$AB = M_1M_2 = x_2 - x_1 \ (\text{or } y_2 - y_1).$$

4°. If only the length of the segment $\overline{AB}$ interests us and its direction is immaterial, then we must take the absolute value of the number obtained from formula (I):

$$\boxed{\overline{AB} = |x_2 - x_1| \quad \text{or} \quad \overline{AB} = |y_2 - y_1|} \qquad \text{(Ia)}$$

5°. If the segment AB is not parallel to either of the coordinate axes, its magnitude is understood to be the length of the segment AB.

Problem. *Find the distance between points* $A(x_1, y_1)$ and $B(x_2, y_2)$.

Solution. From points A and B (Fig. 7) we drop perpendiculars AM_1 and BM_2 onto the axis Ox and perpendiculars AN_1 and BN_2 onto the axis Oy. Producing AN_1 to its intersection with BM_2, we find, from triangle ACB by the Pythagorean theorem,

$$\overline{AB}=\sqrt{\overline{AC}^2+\overline{CB}^2}.$$

But

$$\overline{AC}=\overline{M_1M_2}=|x_2-x_1|,$$
$$\overline{CB}=\overline{N_1N_2}=|y_2-y_1|.$$

Putting these values inside the radical, we obtain

$$\overline{AB}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}. \qquad \textbf{(II)}$$

That is, *the distance between two given points is equal to the square root of the sum of the squares of the differences between like coordinates of these points.*

6°. From formula (II) the distance of the point $A(x, y)$ from the origin $O(0, 0)$ is $\overline{OA}=\sqrt{(x-0)^2+(y-0)^2}$, or

$$\boxed{\overline{OA}=\sqrt{x^2+y^2}} \qquad \textbf{(III)}$$

That is, *the distance of a point from the origin is equal to the square root of the sum of the squares of the coordinates of this point.*

7°. **Examples.** 1. Find the distance between points $A(-4, 3)$ and $B(0, 6)$.

Solution. From formula (III),

$$\overline{AB}=\sqrt{(0+4)^2+(6-3)^2}=\sqrt{16+9}=5.$$

2. Find the point equidistant from points

$$(0, 0),\ (7, -7)\ \text{ and }\ (8, 0).$$

Solution. Let the coordinates of the desired point be equal to x and y. The distance of the point (x, y) from the first of the given points is $\sqrt{x^2+y^2}$, from the second, $\sqrt{(x-7)^2+(y+7)^2}$, and from the third of the given points, $\sqrt{(x-8)^2+y^2}$.

It is given that all these distances are equal. Let us equate the first radical to the second and the third in succession:

$$\sqrt{x^2+y^2}=\sqrt{(x-7)^2+(y+7)^2}$$
$$\sqrt{x^2+y^2}=\sqrt{(x-8)^2+y^2}.$$

After squaring these equations, removing brackets, and combining like terms, we get the system of equations $x-y=7$ and $x=4$. Substituting $x=4$ into the first equation, we find $y=-3$. The required point is $(4, -3)$.

Sec. 4. Dividing a Segment in a Given Ratio

1°. Problem. *The coordinates x_1, y_1 of the point A* (Fig. 8) *and the coordinates x_2, y_2 of the point B being given, find the coordinates x, y of a third point C which divides the segment AB so that the ratio $\frac{AC}{CB}$ is equal to λ.*

Solution. From the points A, B and C draw straight lines AM_1, BM_2 and CM parallel to the axis Oy. These straight lines cut

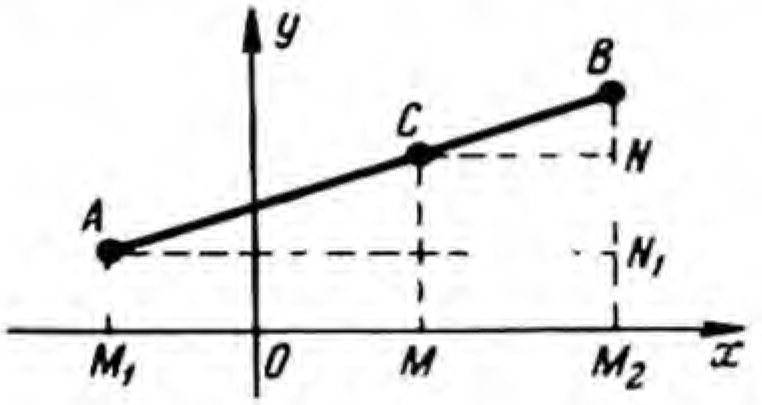

Fig. 8.

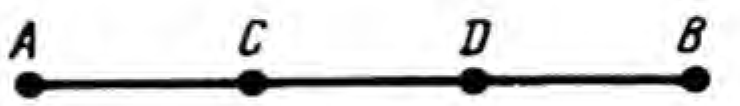

Fig. 9.

the segments AB and M_1M_2 into proportional parts:

$$\frac{M_1M}{MM_2}=\frac{AC}{CB}. \tag{I}$$

But according to formula (I) $M_1M=x-x_1$,

$$MM_2=x_2-x, \quad \text{and} \quad \frac{AC}{CB}=\lambda \text{ (given).}$$

Introducing these values into proportion (I) we have

$$\frac{x-x_1}{x_2-x}=\lambda.$$

Solving this equation for x, we get

$$\boxed{x=\frac{x_1+\lambda x_2}{1+\lambda}} \tag{IV}$$

Drawing from the points A, B and C straight lines parallel to the axis Ox, we find, in analogy with the preceding, that

$$\boxed{y=\frac{y_1+\lambda y_2}{1+\lambda}} \tag{IV}$$

2°. **Example.** Find the points that divide into three equal parts a straight-line segment bounded by the points $A(4, -3)$ and $B(-5, 0)$.

Solution. Let the desired points, reckoning from A to B, be C and D (Fig. 9). Then

$$\frac{AC}{CB}=\frac{1}{2} \quad \text{and} \quad \frac{AD}{DB}=2.$$

This means that when evaluating the coordinates of the points C and D by formulas (IV), we must put λ equal to $\frac{1}{2}$ for point C and equal to 2 for point D. Substituting the given $x_1=4$, $y_1=-3$, $x_2=-5$, $y_2=0$, $\lambda_C=\frac{1}{2}$, $\lambda_D=2$, we get

$$x_C=\frac{4+\frac{1}{2}\cdot(-5)}{1+\frac{1}{2}}=1; \qquad y_C=\frac{-3+\frac{1}{2}\cdot 0}{1+\frac{1}{2}}=-2.$$

$$x_D=\frac{4+2\cdot(-5)}{1+2}=-2; \quad y_D=\frac{-3+2\cdot 0}{1+2}=-1.$$

The required points are $C(1, -2)$, $D(-2, -1)$.

3°. If a point C divides a segment AB into two equal parts, then $AC=CB$ and $\lambda=\frac{AC}{CB}=1$ and formulas (IV) take the form

$$\boxed{x=\frac{x_1+x_2}{2}; \quad y=\frac{y_1+y_2}{2}} \tag{V}$$

Sec. 5. The Angle of a Straight Line with the Axis

1°. **Definition.** *The angle formed by the axis Ox and the straight line AB* (Fig. 10) *is taken to be the angle of counterclockwise*

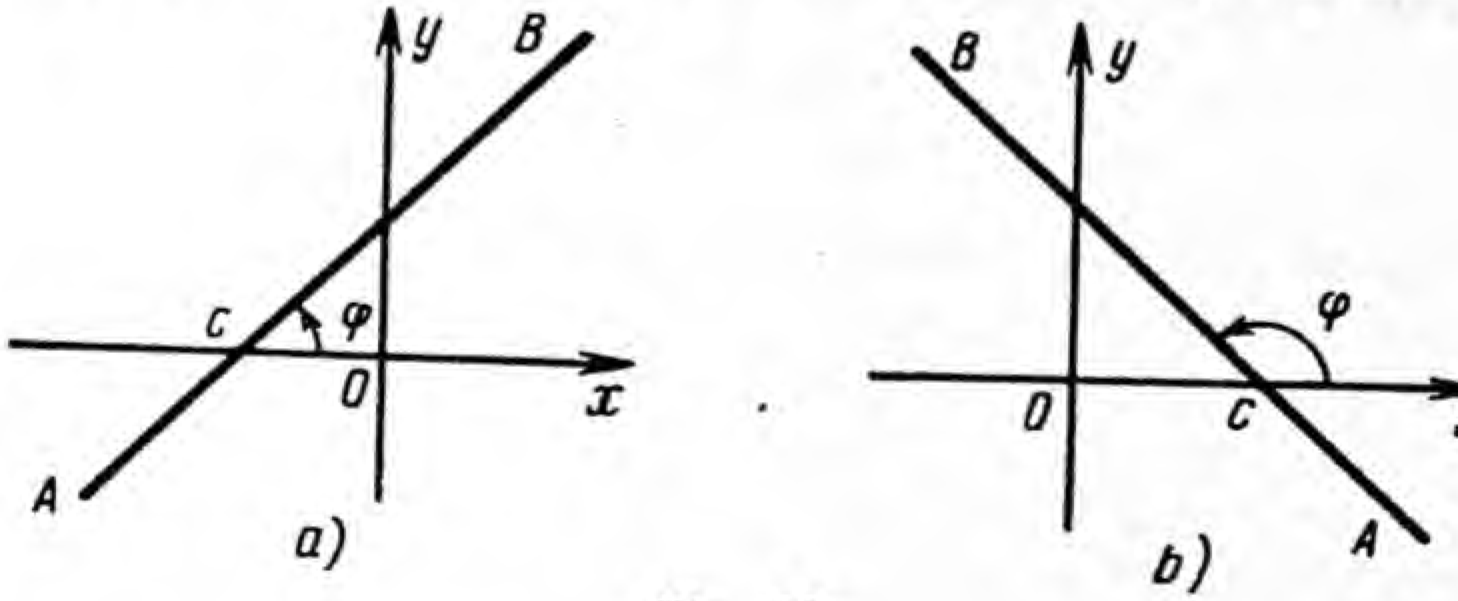

Fig. 10.

rotation of the positive direction of the axis Ox so that the positive direction coincides with the straight line AB.

In Fig. 10a and b, this angle is xCB.

2°. **Problem.** *Find the angle between the straight line passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$, and the axis Ox.*

Solution. Let the straight line AB form with the axis Ox an angle xDB equal to φ (Fig. 11). We shall draw $AC \parallel Ox$ and

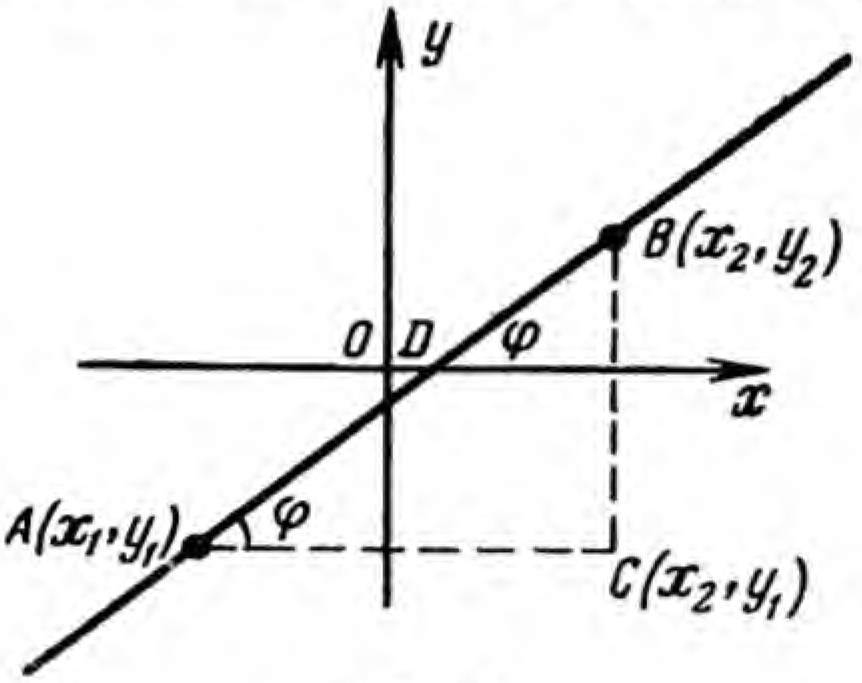

Fig. 11.

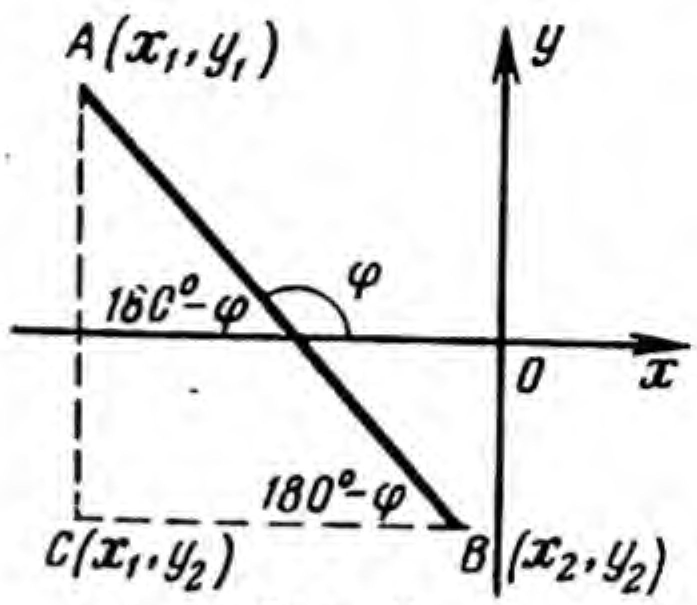

Fig. 12.

$BC \parallel Oy$ to their intersection in point C. The coordinates of C are x_2 and y_1. From the triangle ACB we have

$$\tan \angle CAB = \frac{CB}{AC}. \tag{1}$$

The segment AC, bounded by the points $A(x_1, y_1)$ and $C(x_2, y_1)$ located on a straight line parallel to the axis Ox, is determined by formula (I):

$$AC = x_2 - x_1.$$

Also from formula (I),

$$CB = y_2 - y_1.$$

The angle CAB is equal to φ because $\angle CAB = \angle xDB$ (as corresponding angles).

Introducing the values of $\angle CAB$ and the sides BC and AC into (1), we obtain

$$\boxed{\tan \varphi = \frac{y_2 - y_1}{x^2 - x_1}} \tag{VI}$$

Formula (VI) holds for any location of the points A and B in the plane. For instance, for the location shown in Fig 12, from the triangle ACB we have

$$\tan B = \frac{CA}{CB} = \frac{y_1 - y_2}{x_2 - x_1}.$$

But $\tan B = \tan(180° - \varphi) = -\tan \varphi$, and, therefore, $-\tan \varphi = \frac{y_1 - y_2}{x_2 - x_1}$. Multiplying the left and the right side of the equation by -1, we obtain formula (VI).

3°. **Example.** Find the angle formed by the axis Ox and the straight line connecting the points $A(-2, 1)$ and $B(2, -3)$.

Solution. Putting, in formula (VI), $x_1=-2$, $x_2=2$, $y_1=1$, $y_2=-3$, we obtain

$$\tan\varphi=\frac{-3-1}{2-(-2)}=1,$$

$$\varphi=135°.$$

Sec. 6. Conditions for Parallelism and Perpendicularity

In this section we shall consider straight lines not parallel to the axis Oy.

1°. **Definition.** *The tangent of an angle formed by a straight line with the axis Ox is called the slope of the straight line to the axis Ox.*

2°. If $AB \parallel CD$ (Fig. 13), the angles φ and α formed by the straight lines AB and CD with Ox are equal (as corresponding angles).

Therefore

$$\boxed{\tan\varphi=\tan\alpha} \qquad \text{(VII)}$$

The condition for parallelism of straight lines is that the slopes of the straight lines be equal.

3°. Let $AB \perp BC$ (Fig. 14) and the straight line AB form with the axis Ox an angle equal to φ, and the straight line BC form

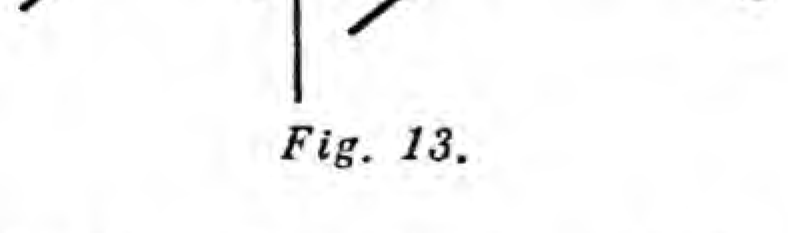

Fig. 13.

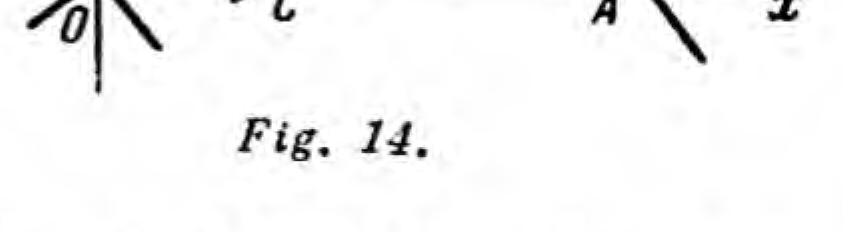

Fig. 14.

with Ox an angle equal to α. Let us draw from the origin O straight lines OA' and OC' parallel, respectively, to AB and BC. Then φ is also the angle through which Ox should be revolved counterclockwise to make it coincide with OA', and α is the angle through which Ox should be revolved to make it coincide

with OC'. Since $OA' \perp OC'$, φ and α differ by $90°$:

$$\varphi - \alpha = 90°.$$

Whence

$$\varphi = 90° + \alpha$$

and

$$\tan\varphi = \tan(90° + \alpha) = -\cot\alpha = -\frac{1}{\tan\alpha}.$$

Therefore,

$$\boxed{\tan\varphi = -\frac{1}{\tan\alpha}} \qquad \text{(VIII)}$$

The condition for perpendicularity of two straight lines is that the slopes of these straight lines be reciprocal in magnitude and opposite in sign.

CHAPTER II

THE STRAIGHT LINE

Sec. 7. A Straight Line as a Locus

1°. **Example.** *Express by an equation the locus of points, in a plane, equidistant from the points A (2, 8) and B (5, 3).*

Solution. It is known from plane geometry that the locus of points equidistant from two given points A and B (Fig. 15) is a straight line MN, perpendicular to the segment AB and passing through its middle point C.

Let us take a point Q on the straight line MN and let its coordinates be (x, y). The distances of Q from the given points A and B are given by formula (II):

$$\overline{AQ} = \sqrt{(x-2)^2 + (y-8)^2},$$

$$\overline{BQ} = \sqrt{(x-5)^2 + (y-3)^2}.$$

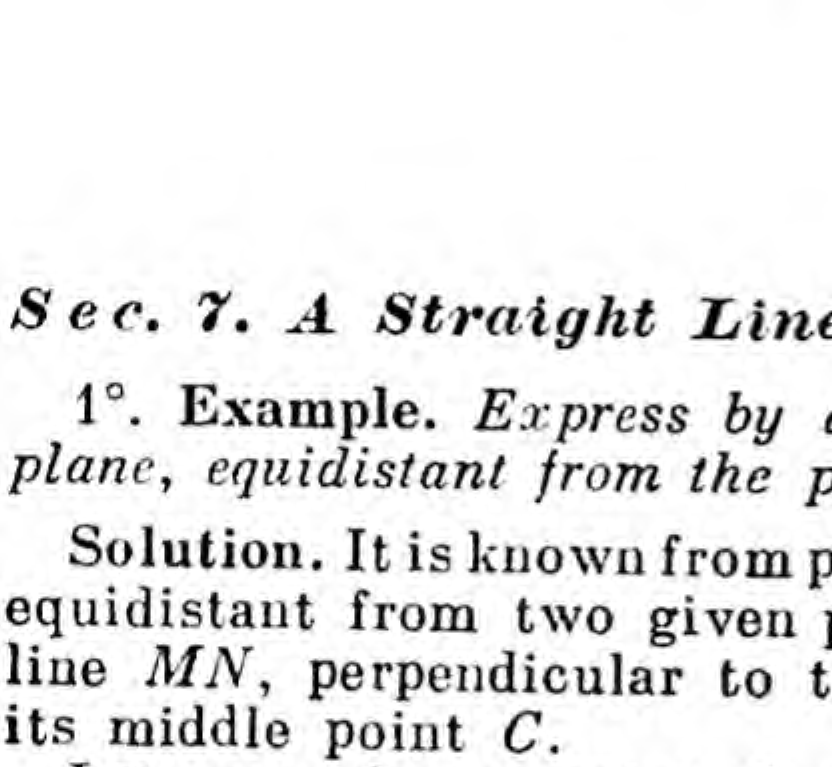

Fig. 15.

And since it is given that these distances are equal, i.e., $\overline{AQ} = \overline{BQ}$, we have

$$\sqrt{(x-2)^2 + (y-8)^2} = \sqrt{(x-5)^2 + (y-3)^2}. \qquad (1)$$

As the point $Q(x, y)$ moves along the straight line MN the values of x and y vary, but they vary in a definite ratio such that equality (1) is not violated. Equality (1) is thus an equation satisfied by the coordinates of all points of the straight line MN without exception. At the same time this equation is not satisfied by the coordinates of any point in the plane not lying on the straight line MN. Indeed, if the point Q happens to lie outside the straight line MN, then it will not be equidistant from the points A and B, $\overline{AQ} \neq \overline{BQ}$ and equality (1) will be violated.

Equating the radicands of (1) and removing the brackets, we have

$$x^2-4x+4+y^2-16y+64=x^2-10x+25+y^2-6y+9,$$
$$\text{or}\quad 3x-5y+17=0. \tag{2}$$

Equation (2), which expresses the condition to be satisfied by the coordinates x, y of points in the plane in order that they should belong to the straight line MN, is called *the equation of the straight line MN.*

Examples. 1. In order to find out whether the point P (2, 4) belongs to the straight line MN, we substitute in equation (2) x by 2 and y by 4. We then obtain

$$3\cdot 2-5\cdot 4+17=3\neq 0.$$

The coordinates of the point P do not satisfy the necessary condition (that the points belong to the straight line MN) and, consequently, the point P does not lie on the straight line MN.

2. Point P $(-4, 1)$ belongs to the straight line MN because its coordinates $(-4, 1)$ satisfy equation (2):

$$3\cdot(-4)-5\cdot 1+17=0,\ 0=0.$$

2°. The coordinates of an arbitrary point (x, y) of a given locus are called *current (moving) coordinates. The equation of the line expresses an unchanging relation between the moving coordinates of its point.*

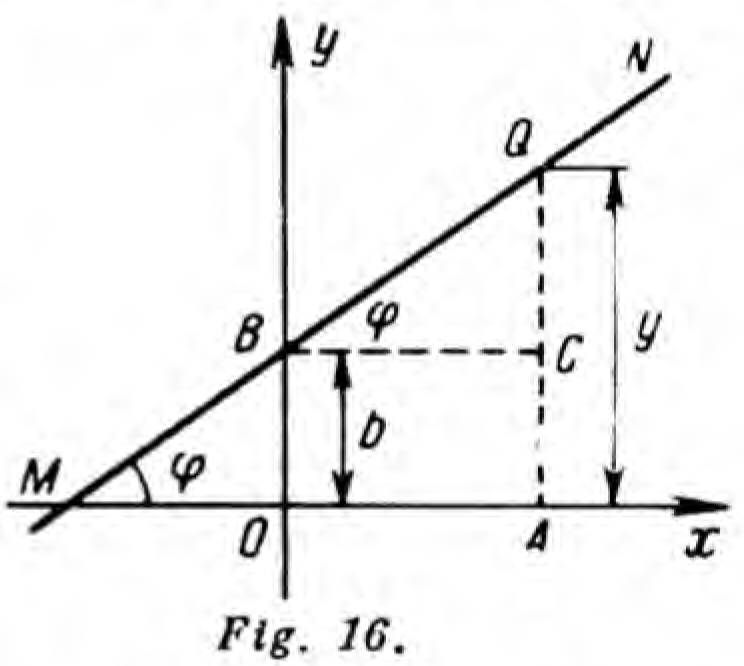

Fig. 16.

Sec. 8. Equation of a Straight Line (Slope-Intercept Form)

1°. Theorem. *Any straight line can be expressed by an equation of the first degree between the moving coordinates.*

Proof. Case 1. The straight line MN (Fig. 16) is not parallel to the axis Oy. For an arbitrary point Q (x, y) of the straight line MN we have the equation

$$AQ=AC+CQ, \tag{1}$$

where $AQ=y$, $AC=OB$, i.e., AC is equal to the ordinate of the point of intersection of MN with the axis Oy. We assume that

$$AC=OB=b.$$

From the right-angled triangle BCQ we have

$$CQ=BC\cdot\tan\varphi,$$

where φ is the angle formed by the straight line MN and the axis Ox, since

$$\varphi = \angle xMN = \angle CBQ$$

and $BC = OA = x$.

Putting $\tan \varphi = k$, we obtain

$$CQ = k \cdot x.$$

Introducing the values of AQ, AC and CQ into equation (1), we get

$$\boxed{y = kx + b} \qquad \text{(IX)}$$

that is, a first-degree equation in the current coordinates x and y.

The coordinates $(0, b)$ of the point of intersection (B) of the straight line M with the axis Oy also satisfy this equation.

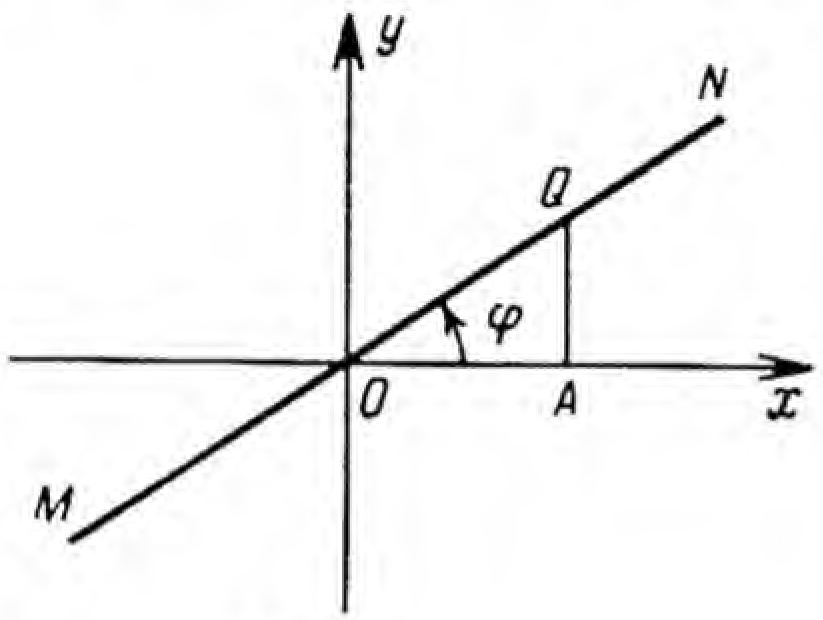

Fig. 17.

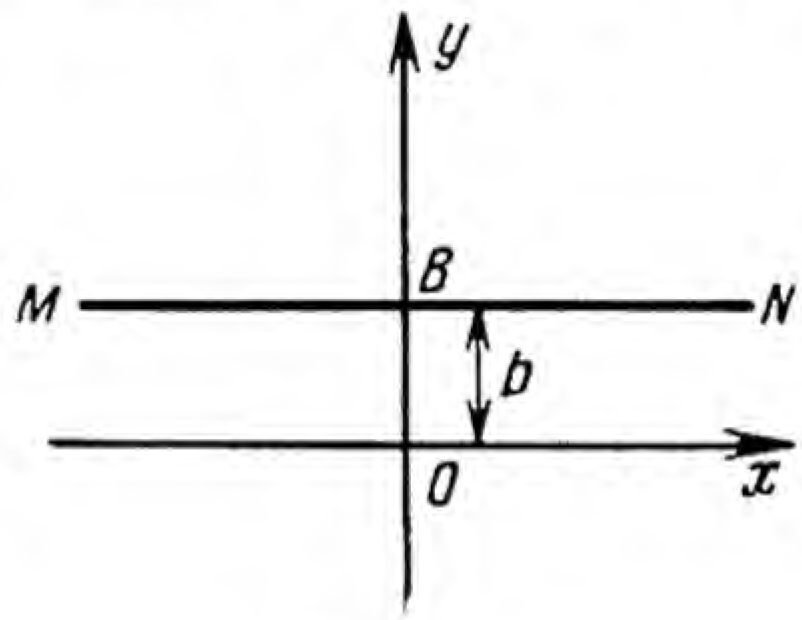

Fig. 18.

Indeed, replacing, in equation (IX), x by zero and y by the number b, we obtain the identity

$$b = b.$$

There can be three special cases here.

1. If the straight line MN passes through the origin O (Fig. 17), the ordinate of the point of its intersection with the axis Oy, $b = 0$, and equation (IX) takes the form

$$\boxed{y = kx} \qquad \text{(X)}$$

It is satisfied by the coordinates of all the points of MN, including the origin (O, O).

2. If the straight line MN is parallel to the axis Ox (Fig. 18), i.e., angle $\varphi = 0$ and $k = \tan \varphi = 0$, equation (IX) takes the form

$$\boxed{y = b} \qquad \text{(XI)}$$

Equality (XI) does not contain x. Nevertheless it expresses the relation between x and y: for each value of x, i.e., for each point of a straight line parallel to Ox, the ordinate is equal to b; and, conversely, in a plane every point whose ordinate is equal to b belongs to this straight line.

3. If the straight line coincides with the axis Ox, then by introducing $b=0$ into equation (XI) we obtain the equation of the axis:

$$\boxed{y=0} \qquad \text{(XII)}$$

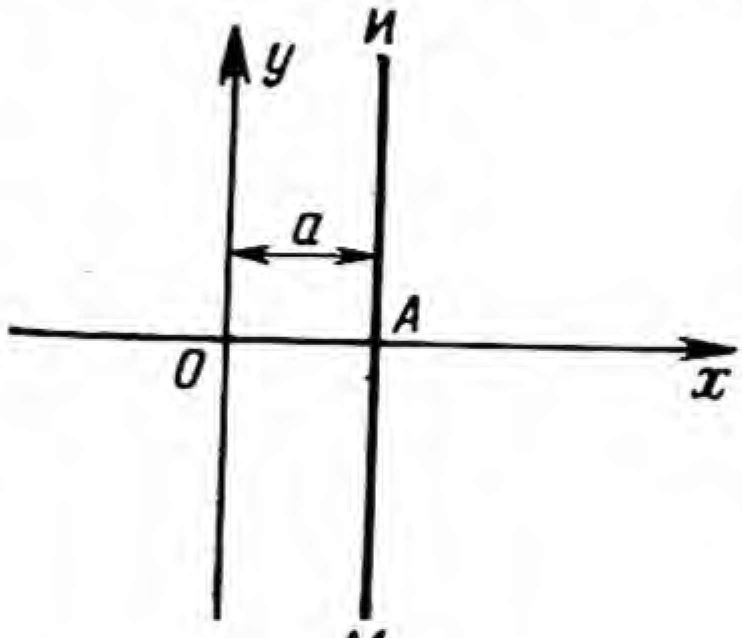

Fig. 19.

Case 2. The straight line MN is parallel to the axis Oy (Fig. 19).

Let the abscissa of the point of its intersection with the axis Ox be equal to a. The equation of such a straight line is

$$\boxed{x=a} \qquad \text{(XIII)}$$

Indeed, the abscissa of each of the points of a straight line parallel to the axis Oy (i.e., for each value of y) is equal to a and, conversely, in a plane every point whose abscissa is equal to a belongs to this straight line.

If the straight line coincides with the axis Oy, $a=0$, then from equation (XIII) we obtain the equation of the axis Oy:

$$\boxed{x=0} \qquad \text{(XIV)}$$

Thus any straight line is expressed by a first-degree equation in the moving coordinates.

2°. In equation (IX), k is the *slope of the straight line* and $y=kx+b$ is called *slope-intercept form of the equation of a straight line*. The absolute term b of this equation is called the y-intercept or intercept on the y-axis because when $x=0$, $y=b$ (from equation IX). The slope k and the y-intercept b are constant values for the given straight line and are the *parameters* of the equation of the straight line. In general, we call parameters those constants that characterise an object and distinguish it from other objects homogeneous with it.

Sec. 9. General Form of the Equation of a Straight Line and Its Special Cases

1°. **Converse Theorem.** *Any equation of the first degree between the coordinates x and y represents a straight line.*

Proof. An equation of the first degree in x, y is of the form

$$\boxed{Ax+By+C=0} \tag{XV}$$

Let $B \neq 0$. Solving equation (XV) for y, we get

$$y=-\frac{A}{B}x-\frac{C}{B}.$$

Comparing this expression with equation (IX), we have

$$\boxed{-\frac{A}{B}=k=\tan\varphi} \tag{XVI}$$

$$\boxed{-\frac{C}{B}=b} \tag{XVII}$$

That is, the equation $Ax+By+C=0$, when $B \neq 0$, represents a straight line with slope $k=-\frac{A}{B}$ and y-intercept $b=-\frac{C}{B}$.

In particular:

1) $C=0$. Equation (XV) takes the form

$$\boxed{Ax+By=0} \tag{XVIII}$$

and represents a straight line which passes through the coordinate origin, since the coordinates of the origin (O, O) satisfy equation (XVIII).

2) $A=0$. Equation (XV) takes the form

$$\boxed{By+C=0} \tag{XIX}$$

Solving it for y,

$$y=-\frac{C}{B}, \quad \text{or} \quad y=b,$$

we come to the conclusion (formula XI) that it represents a straight line parallel to the axis of abscissas.

3) $A=0$ and $C=0$. In this case equation (XV) takes the form

$$By=0 \quad \text{or} \quad y=0$$

and represents the axis Ox (formula XII).

4) If the coefficient $B=0$, equation (XV) takes the form

$$\boxed{Ax+C=0} \tag{XX}$$

Solving this for x,

$$x=-\frac{C}{A}=a,$$

we come to the conclusion (formula XIII) that it represents a straight line parallel to the axis of ordinates.

But if in equation (XX) $C=0$, $A \neq 0$, then it has the form

$$Ax=0 \quad \text{or} \quad x=0$$

and represents the y-axis (formula XIV).

Thus, any equation of the first degree between the coordinates x and y represents a straight line.

2°. Equation (XV)

$$Ax+By+C=0$$

is called the general form of the equation of a straight line. Equations (XVIII-XX) are its special cases.

Sec. 10. Equation of a Straight Line (Intercept Form)

The position of a straight line with respect to a system of coordinate axes can be determined by the segments $ON=a$ and $OM=b$ (Fig. 20) which are cut off by the given straight line MN on the axes Ox and Oy respectively.

From the similarity of triangles AQN and OMN (Fig. 20), we have

$$\frac{AQ}{OM}=\frac{AN}{ON} \quad \text{or}$$

$$\frac{y}{b}=\frac{a-x}{a}.$$

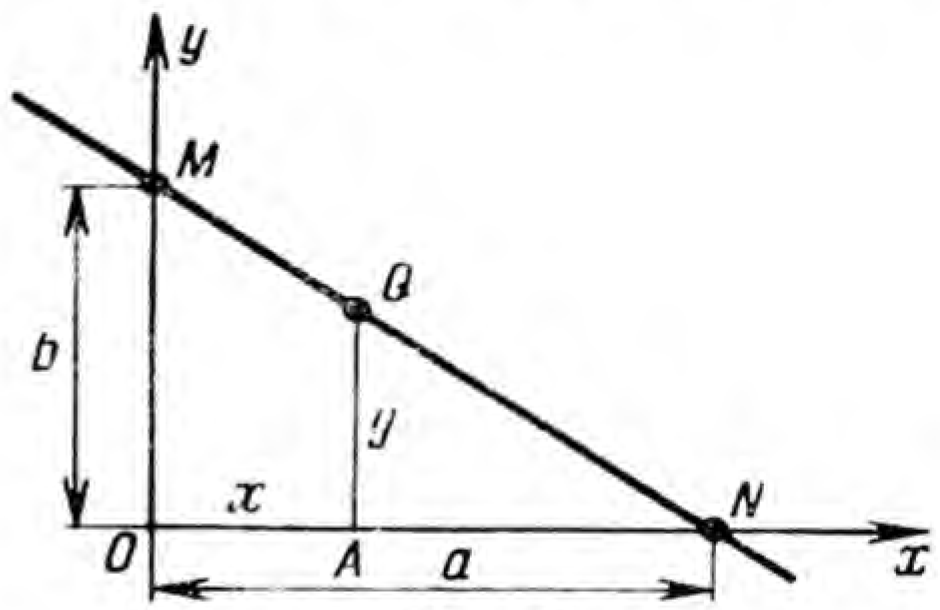

Fig. 20.

Dividing $a-x$ by a termwise, we obtain

$$\frac{y}{b}=1-\frac{x}{a} \quad \text{or}$$

$$\boxed{\frac{x}{a}+\frac{y}{b}=1} \qquad \text{(XXI)}$$

Equation (XXI) is called the intercept form of the equation of a straight line.

Sec. 11. Solved Examples

1°. Find the equation of the straight line which cuts off a segment equal to 3 on the axis Oy and makes an angle $\varphi=120°$ with the axis Ox.

Solution. To obtain the equation of the given straight line it is necessary to find the numerical values of the parameters k and b and introduce them into the equation:

$$y=kx+b.$$

It follows from the statement of the problem that

$$b=3,$$

$$k=\tan 120°=\tan (180°-60°)=-\tan 60°=-\sqrt{3}.$$

Consequently, the required equation is

$$y = -x\sqrt{3} + 3.$$

2°. Determine the slope and the y-intercept of the straight line represented by the equation $2x - 3y - 6 = 0$.

Solution. 1st procedure. We solve the equation for y, and obtain an equation in slope-intercept form:

$$y = \frac{2}{3}x - 2,$$

from which we conclude that the slope $k = \frac{2}{3}$ and the y-intercept $b = -2$.

2nd procedure. With $A = 2$, $B = -3$, $C = -6$, we find, from (XVI) and (XVII),

$$k = -\frac{A}{B} = -\frac{2}{-3} = \frac{2}{3},$$

$$b = -\frac{C}{B} = -\frac{-6}{-3} = -2.$$

3°. Find out whether there are straight lines parallel or mutually perpendicular among those represented by the equations: $3x - 2y + 1 = 0$, $6x - 4y - 5 = 0$ and $2x + 3y - 7 = 0$.

Solution. The slopes of parallel straight lines are equal; those of mutually perpendicular straight lines are reciprocal in magnitude and opposite in sign (Sec. 6). From formula (XVI) we find the slopes: for the first straight line $k_1 = \frac{3}{2}$, for the second straight line $k_2 = \frac{3}{2}$ and for the third, $k_3 = -\frac{2}{3}$.

The first and the second straight lines are parallel, since $k_1 = k_2$, the third straight line is perpendicular to the first two, since

$$k_3 = -\frac{1}{k_1} = -\frac{1}{k_2}.$$

4°. Form the equation of a straight line that cuts off on the axes Ox and Oy segments -3 and $\frac{1}{2}$, respectively.

Solution. It is given that $a = -3$, $b = \frac{1}{2}$. Substituting these values into equation (XXI) we get the desired equation:

$$\frac{x}{-3} + \frac{y}{\frac{1}{2}} = 1 \quad \text{or} \quad x - 6y + 3 = 0.$$

5°. What is the equation of the straight line that has equal x- and y-intercepts and passes through the point $(-1, -3)$?

Solution. It is given that $b = a$ in equation (XXI), and therefore the equation takes the form

$$\frac{x}{a} + \frac{y}{a} = 1 \quad \text{or} \quad x + y = a.$$

The point $(-1, -3)$ belongs to the straight line $x+y=a$. Therefore, if in the equation $x+y=a$ we substitute the values -1, -3 for the current coordinates x, y, the equation will hold:

$$-1-3=a.$$

Hence $a=-4$ and the sought-for equation of the straight line is

$$x+y=-4 \text{ or } x+y+4=0.$$

6°. Reduce the equation $3x-4y+2=0$ to the intercept form.

Solution. Transpose the absolute term 2 to the right-hand side of the equation,

$$3x-4y=-2,$$

and divide both sides of the equation by -2:

$$-\frac{3x}{2}+2y=1.$$

We then get the intercept form of the equation of a straight line:

$$\frac{x}{-\frac{2}{3}}+\frac{y}{\frac{1}{2}}=1$$

and the x- and y-intercepts are $a=-\frac{2}{3}$, $b=\frac{1}{2}$.

Sec. 12. Construction of a Straight Line When Its Equation Is Given

1°. In order to construct a straight line it is sufficient to locate any two of its points by means of coordinates. Given the equation, one can most easily define the coordinates of the points of intersection of the straight line with the coordinate axes. Putting $y=0$, we find from the given equation of the straight line the abscissa of the point of its intersection with the axis Ox; putting $x=0$, we find from the given equation the ordinate of the point of intersection of the straight line with the axis Oy.

Example. Construct the straight line defined by the equation

$$3x+2y+6=0.$$

Solution. Assuming in this equation

1) $y=0$, we have $3x+6=0$, $x=-2$;
2) $x=0$, we have $2y+6=0$; $y=-3$.

Marking points A $(-2, 0)$ and B $(0, -3)$ (Fig. 21), draw a straight line through them with the help of a ruler.

2°. **Example.** Construct a straight line in accordance with the equation

$$3x-2y=0.$$

Solution. This equation of the straight line does not contain an absolute term and, therefore, the straight line passes through

the origin. Let us assign to x some value other than zero, say 2, and introduce it into the given equation. We obtain

$$3\cdot 2-2y=0,\ y=3.$$

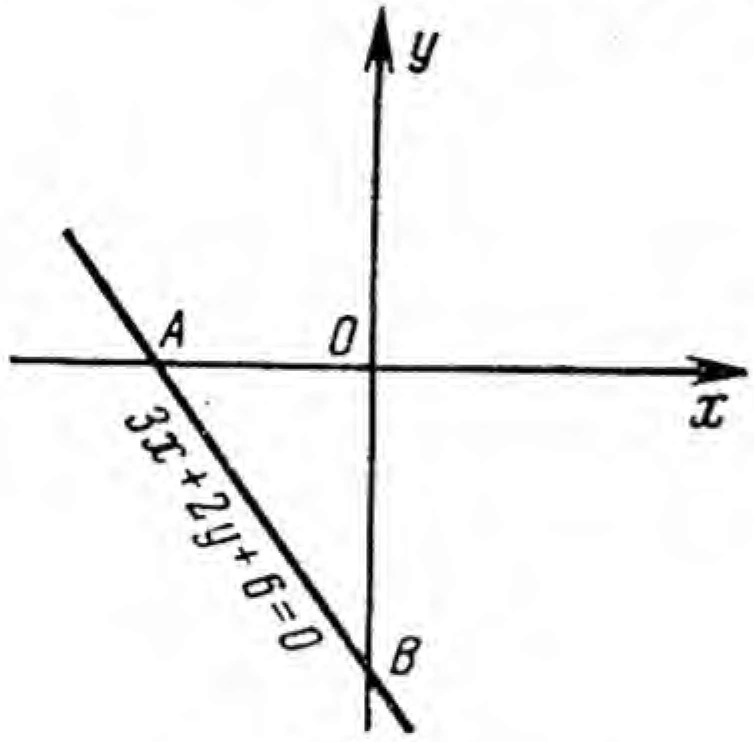

Fig. 21.

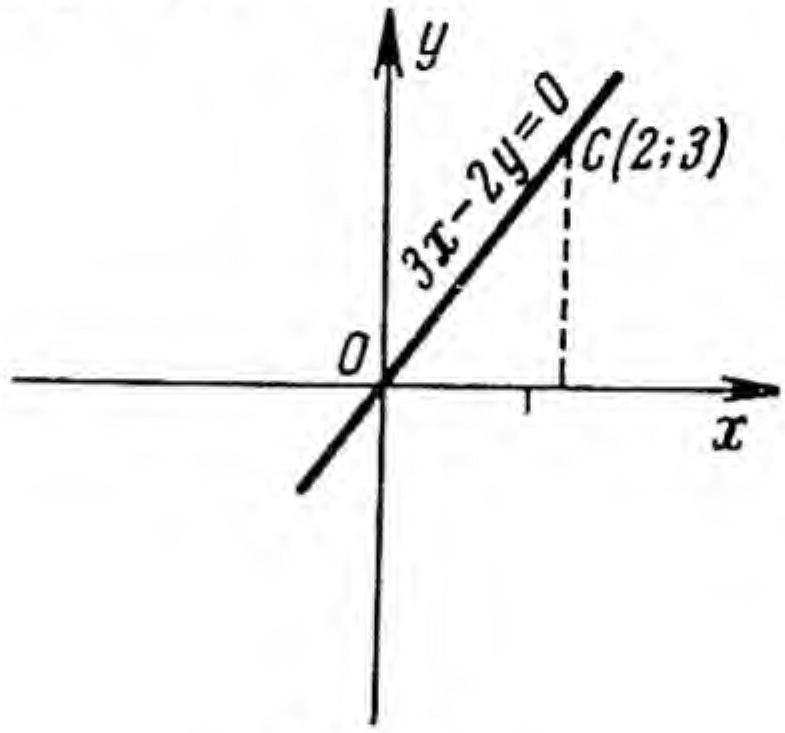

Fig. 22.

Having constructed a point C (2, 3) (Fig. 22) we use a ruler and draw throuph C and the origin O a straight line CO. This will be the straight line defined by the equation

$$3x-2y=0.$$

3°. If the equation of the straight line is $y=b$ or $x=a$ then the construction of the line is reduced to drawing through the given point (0, b) or (a, 0) a straight line parallel to the axis Ox or Oy, respectively.

Sec. 13. The Point of Intersection of Two Straight Lines

1°. There are three possible cases of the mutual positions in a plane of the straight lines

$$Ax+By+C=0 \qquad (1)$$

and

$$A'x+B'y+C'=0: \qquad (2)$$

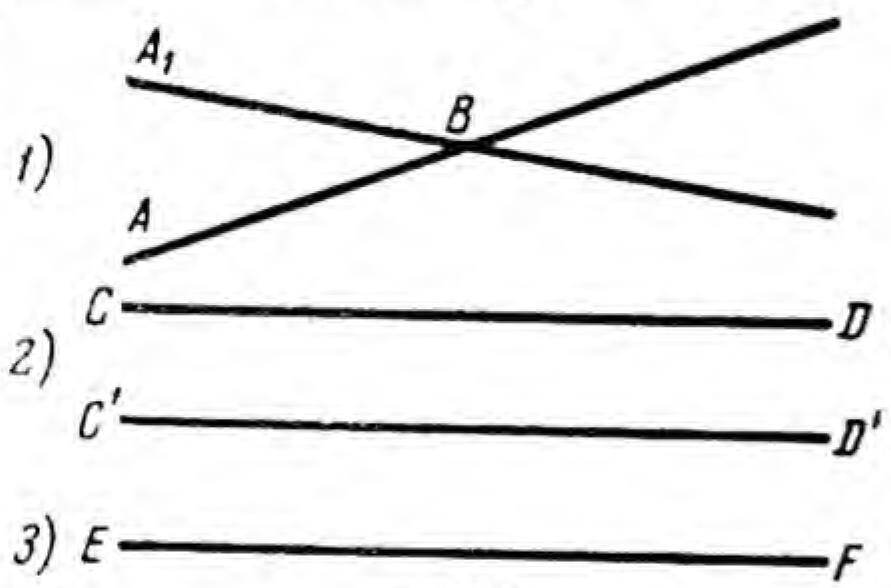

Fig. 23.

1) the straight lines have one point in common (they intersect) (Fig. 23, *1*);

2) the straight lines have no points in common (they are parallel (Fig. 23, *2*);

3) the straight lines have an infinite number of points in common (they coincide) (Fig. 23, *3*).

Let us solve the problem: how to find out, from the given equations, which of the three cases occurs?

2°. *Equations (1) and (2) represent one and the same straight line if their coefficients are proportional.*

Indeed, if

$$\frac{A}{A'}=\frac{B}{B'}=\frac{C}{C'}=\lambda, \text{ then } A=\lambda A',\ B=\lambda B',\ C=\lambda C'$$

and equation (1) may be written as

$$\lambda(A'x+B'y+C')=0. \tag{3}$$

All the values of the coordinates (x, y) which satisfy equation (2) also satisfy equation (3). Indeed, since coefficient $\lambda \neq 0$ in equation (3), $A'x+B'y+C'=0$. But this is equation (2). Therefore, each point (x, y) of the straight line represented by (2) belongs also to the straight line represented by (3). Consequently these straight lines coincide.

3°. *Equations (1) and (2) represent parallel straight lines if and only if the coefficients of the moving coordinates are proportional.*

Proof. At least one of the absolute terms C or C' must be different from zero, otherwise the straight lines $Ax+By=0$ and $A'x+B'y=0$ coincide if their coefficients A, B, A', B' are proportional.

If $\frac{A}{A'}=\frac{B}{B'}=\lambda$, then by dividing equation (1) by λ we have

$$A'x+B'y+\frac{C}{\lambda}=0.$$

The difference between the left-hand sides of the equations

$$A'x+B'y+\frac{C}{\lambda}=0$$

and

$$A'x+B'y+C'=0$$

is different from zero. Therefore, there is not a single number pair (x, y) which could satisfy both equations. Hence, these straight lines have no points in common, they are parallel.

Conversely, if straight lines are parallel, their slopes are equal:

$$k=k',$$

but since $k=-\frac{A}{B}$, $k'=-\frac{A'}{B'}$ it follows that $\frac{A}{B}=\frac{A'}{B'}$. That is, the coefficients of the moving coordinates in our equations are proportional.

4°. *The straight lines*

$$Ax+By+C=0, \tag{1}$$

$$A'x+B'y+C'=0 \tag{2}$$

***intersect** if $AB'-A'B\neq 0$.*

Indeed, multiplying (1) by B' and (2) by B and subtracting the second product from the first, we have

$$\begin{array}{r} -\begin{array}{r} AB'x+BB'y+CB'=0 \\ A'Bx+BB'y+C'B=0 \end{array} \\ \hline (AB'-A'B)\,x+(CB'-C'B)=0. \end{array} \tag{4}$$

Multiplying (1) by A' and (2) by A and subtracting the first product from the second, we have

$$\begin{array}{r} -\begin{array}{r} AA'x+A'By+A'C=0 \\ AA'x+AB'y+AC'=0 \end{array} \\ \hline (AB'-A'B)\,y+(AC'-A'C)=0. \end{array} \tag{5}$$

Equations (4) and (5) have definite solutions only when the coefficient of the unknowns x and y

$$AB'-A'B\neq 0. \tag{6}$$

These solutions define the point that belongs to both straight lines, i. e. the point of intersection of the straight lines (1) and (2).

The coordinates of the point of intersection of the straight lines (1) and (2) are

$$\boxed{x=-\frac{CB'-C'B}{AB'-A'B}, \quad y=-\frac{AC'-A'C}{AB'-A'B}} \tag{XXII}$$

5°. The denominator in equations (XXII) is expression (6), which by the statement of the problem is not equal to zero (division by zero being impossible).

When $AB'-A'B=0$ and the absolute terms of equations (4) and (5) ($CB'-C'B$ and $AC'-A'C$) are different from zero, we have

$$AB'=A'B \quad \text{or} \quad \frac{A}{A'}=\frac{B}{B'},$$

i. e., straight lines (1) and (2) are parallel because the coefficients of the unknowns x and y are proportional.

When both the coefficient of the unknown and the absolute terms in equations (4) and (5) are equal to zero,

$$AB'-A'B=0, \quad CB'-C'B=0, \quad AC'=A'C=0,$$

we have, from these equations,

$$\frac{A}{A'}=\frac{B}{B'}; \quad \frac{B}{B'}=\frac{C}{C'}; \quad \frac{C}{C'}=\frac{A}{A'}. \quad \text{Whence} \quad \frac{A}{A'}=\frac{B}{B'}=\frac{C}{C'}.$$

That is, equations (1) and (2) represent one straight line.

6°. It is clear from the foregoing that the *nature of the mutual arrangement of two straight lines is made clear by the simultaneous solution of the equations of these lines.*

7°. **Numerical examples.** 1. Find the point of intersection of the straight lines

$$3x+4y-18=0 \text{ and } 4x-3y+1=0.$$

Solution.

$$\begin{array}{l|c|c|} 3x+4y-18=0 & 3 & 4 \\ 4x-3y+1=0 & 4 & 3 \end{array} + \quad \begin{array}{r} 9x+12y-54=0; \\ 16x-12y+\ 4=0; \\ \hline 25x-50=0; \\ x=2; \end{array} - \quad \begin{array}{r} 12x+16y-72=0 \\ 12x-\ 9y+\ 3=0 \\ \hline 25y-75=0 \\ y=3. \end{array}$$

Answer. The point of intersection of the given straight lines is (2, 3).
2. Find the point of intersection of the straight lines

$$2x-7y=3 \text{ and } 14y-4x=1.$$

Solution.

$$\begin{array}{l|c|} 2x-7y=3 & 2 \\ 14y-4x=1 & 1 \end{array} + \quad \begin{array}{r} 4x-14y=6 \\ 14y-\ 4x=1. \\ \hline 0=7. \end{array}$$

The system is incompatible. The coefficients of the current coordinates of the given equations are proportional, the straight lines are parallel.
3. Find the point of intersection of the straight lines $x-3y+2=0$ and $3x-9y+6=0$.
Solution. Multiplying the first equation by 3 and subtracting from the resulting product the second of the given equations, we obtain

$$0=0,$$

which shows the solution to be indeterminate. The given equations are equivalent and represent one straight line because their coefficients are proportional.

Sec. 14. Equation of a Straight Line Passing Through the Point (x_1, y_1) in a Given Direction

1°. A straight line passes through the given point (x_1, y_1) and makes with the axis Ox an angle φ. Let us form its equation.
It is given that in

$$y=kx+b \tag{1}$$

the parameter $k=\tan\varphi$ is a known number. The straight line (1) passes through the point (x_1, y_1); therefore, x_1, y_1 satisfy equation (1) and, replacing the moving coordinates x, y by the given values x_1, y_1, we get the following equation:

$$y_1=kx_1+b. \tag{2}$$

From this we can find the value of b and introduce it into equation (1). But it is easier to eliminate b from equation (1)

by subtracting equation (2) from (1). This subtraction gives the equation of a straight line passing through a given point (x_1, y_1) in a given direction:

$$\boxed{y-y_1=k(x-x_1)} \tag{XXIII}$$

2°. Equation (XXIII), in which k can have an infinite number of values, is the *equation of a pencil of straight lines with its centre in the point (x_1, y_1).*

3. **Example.** Draw a straight line through the point of intersection of the straight lines $2x-3y+7=0$ and $5x+y+9=0$ such that it is perpendicular to the straight line $2x-y+1=0$.

Solution. "To draw a straight line" means "to write the equation of that straight line". Let us first find the coordinates of the point of intersection of the given straight lines by solving the system of given equations. The point of intersection of the straight lines has coordinates $(-2, 1)$.

Now we determine the slope k of the straight line passing through the point $(-2, 1)$ perpendicular to the straight line $2x-y+1=0$. The slope of the straight line $2x-y+1=0$ is $k_1=2$, that of the straight line perpendicular to it is $k=-\frac{1}{2}$ (see Sec. 6).

By putting, in equation (XXIII), $x_1=-2$, $y_1=1$ and $k=-\frac{1}{2}$, we obtain the sought-for equation

$$y-1=-\frac{1}{2}(x+2) \text{ or } x+2y=0.$$

Sec. 15. Equation of a Straight Line Passing Through Two Given Points (x_1, y_1) and (x_2, y_2)

1°. A straight line passing through the given points (x_1, y_1) and (x_2, y_2) has, by virtue of (VI), a slope equal to

$$\frac{y_2-y_1}{x_2-x_1}. \tag{1}$$

Let us take the equation of a pencil of straight lines whose centre is in the point (x_1, y_1),

$$y-y_1=k(x-x_1),$$

and assign to the slope k the value (1). Then the equation

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \tag{2}$$

will represent that particular straight line of the pencil of straight lines with centre in point (x_1, y_1) which passes through point (x_2, y_2). Dividing equation (2) by y_2-y_1, let us represent the

equation of the straight line which passes through the two given points (x_1, y_1) and (x_2, y_2) in the form

$$\boxed{\frac{y-y_1}{y_2-y_1}=\frac{x-x_2}{x_2-x_1}} \qquad \text{(XXIV)}$$

2°. Write the equation of a straight line that passes through the points A (1, 4) and B (—3, 2).

Solution. If in equation (XXIV) we put $x_1=1$, $y_1=4$, $x_2=-3$, $y_2=2$, we shall obtain the equation we need:

$$\frac{y-4}{2-4}=\frac{x-1}{-3-1}$$

Or, multiplying by -4, we get

$$2(y-4)=x-1 \text{ or } x-2y+7=0.$$

3°. If a straight line passes through points A (2, 3) and B (—2, 3), then by formula (XXIV) we get

$$\frac{y-3}{0}=\frac{x-2}{-4}\,.$$

But division by zero is impossible. We note that different points of the straight line AB have the same ordinate $y=3$. Therefore AB is parallel to the axis Ox and its equation is

$$y=3.$$

4°. Find the condition under which three given points lie on one straight line.

Solution. Let the given points be (x_1, y_1), (x_2, y_2) and (x_3, y_3). The equation of the straight line which passes through the first two points is

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}\,.$$

In order that the third given point (x_3, y_3) should also lie on this straight line, it is sufficient for its coordinates to satisfy the written equation, i. e., for the following to obtain:

$$\boxed{\frac{y_3-y_1}{y_2-y_1}=\frac{x_3-x_1}{x_2-x_1}}$$

Sec. 16. The Angle Between Two Straight Lines

1°. The phrase "the angle between the straight lines AB and BC" will be used to denote "the angle that the straight line BC forms with the straight line AB".

Definition. *The angle formed by a certain straight line BC with the straight line AB* (Fig. 24) *is taken to mean the angle through which the straight line AB must be revolved counterclockwise about point B so that AB should coincide with the straight line BC.*

2°. Let the straight lines AB and BC be represented by the equations

$$y = k_1 x + b_1, \qquad (AB)$$

and

$$y = k_2 x + b_2 \qquad (BC).$$

Let the straight line AB and the axis Ox form an angle equal to φ_1, and the straight line BC and the axis Ox an angle equal

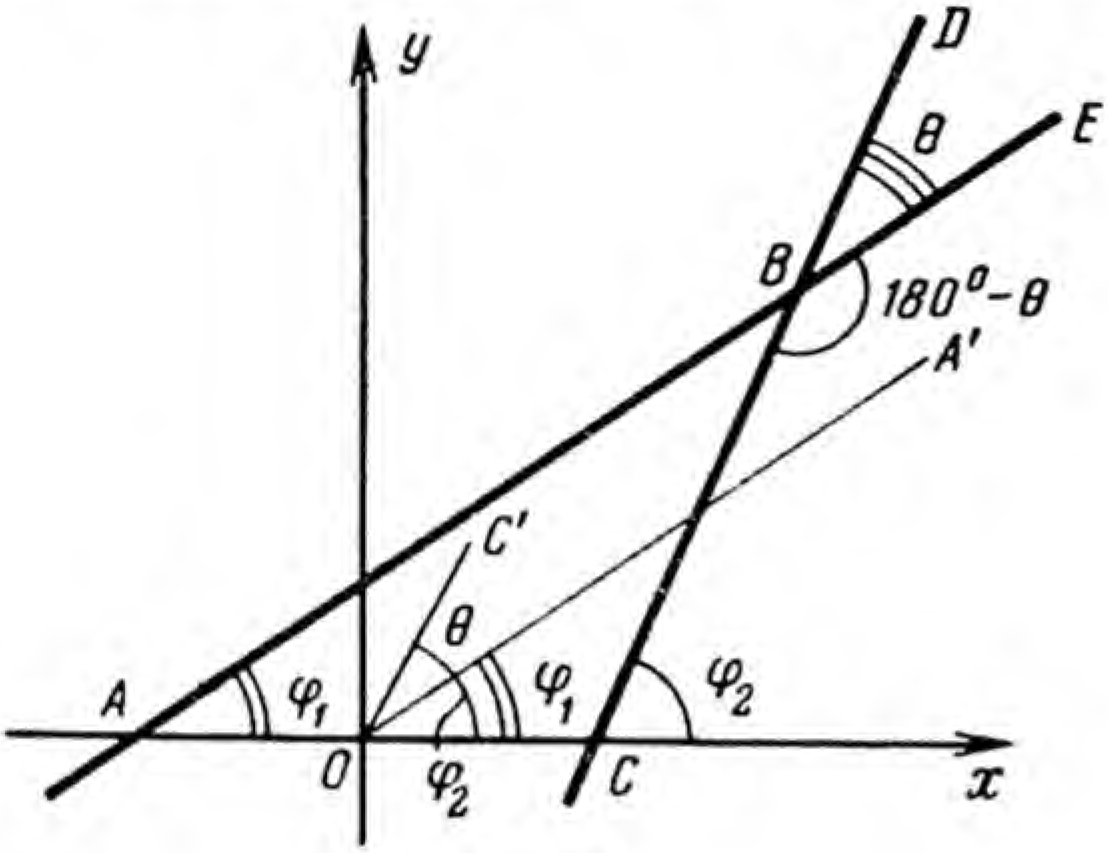

Fig. 24.

to φ_2 (Fig. 24). The angle between AB and BC we shall denote by Θ. Draw from the origin O straight lines OA' and OC' parallel respectively to AB and BC. Since φ_1 is the angle through which Ox must be revolved counterclockwise so that it coincides with OA', and φ_2 is the angle through which Ox must be similarly revolved so that it coincides with OC', it is evident that the angle Θ between OA' and OC' is equal to the difference $\varphi_2 - \varphi_1$.

$$\Theta = \varphi_2 - \varphi_1.$$

But if the angles are equal, their tangents are also equal:

$$\tan \Theta = \tan(\varphi_2 - \varphi_1).$$

Or (by trigonometry for the tangent of the difference between two angles),

$$\tan \Theta = \frac{\tan \varphi_2 - \tan \varphi_1}{1 + \tan \varphi_1 \cdot \tan \varphi_2}.$$

But $\tan \varphi_1 = k_1$, $\tan \varphi_2 = k_2$, therefore

$$\boxed{\tan \Theta = \frac{k_2 - k_1}{1 + k_1 k_2}} \qquad \text{(XXV)}$$

The angle Θ is acute if $\tan\Theta$ is positive, and it is obtuse if $\tan\Theta$ is negative; $\Theta=90°$ when $1+k_1k_2=0$ and $\Theta=0$ when $k_2-k_1=0$ (Sec. 6).

We note that the angle between BC and AB ($\angle CBE$) is an adjacent one for Θ and is therefore equal to $180°-\Theta$. Since $\tan(180°-\Theta)=-\tan\Theta$,

$$\tan(180°-\Theta)=\frac{k_1-k_2}{1+k_1\cdot k_2}.$$

Sometimes it is necessary to find the value of the *acute* angle between the straight lines. In this case we must take the absolute value of $\tan\Theta$ as defined by formula (XXV).

3°. Examples. 1. Find the angle between the straight lines $y=2x-3$ and $y=5x+1$.

Solution. From the equations written above it follows that

$$k_1=2,\ k_2=5.$$

By formula (XXV)

$$\tan\Theta=\frac{5-2}{1+5\cdot 2}=\frac{3}{11}=0.2727.$$

$$\Theta=15°15'.$$

2. Find the equation of the straight line which passes through the origin and forms an angle of 45° with the straight line $y=3x+5$.

Solution. By the statement of the problem (Fig. 25), $\Theta=45°$; and the slope of the given straight line is k_1, inasmuch as the statement of the

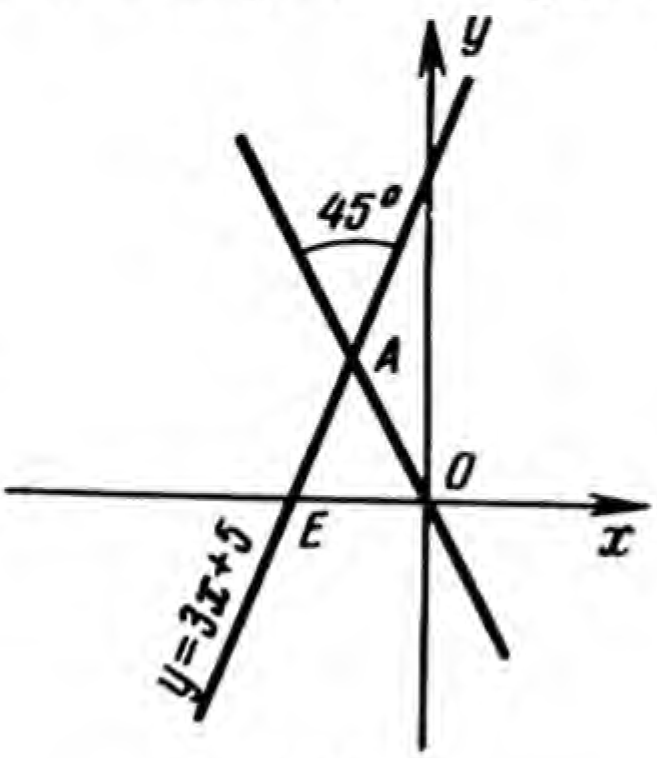

Fig. 25.

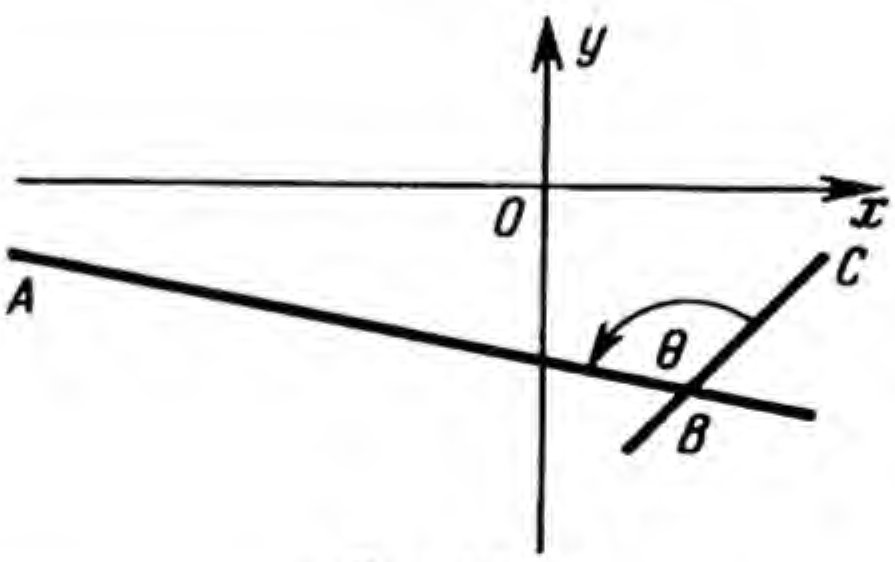

Fig. 26.

problem is such that in order to obtain an angle of 45° we must revolve the *given* straight line counterclockwise to make it coincide with the straight line desired:

$$\tan\Theta=\tan 45°=1;\ k_1=3.$$

Let us introduce these values into formula (XXV) and find k_2.

$$1=\frac{k_2-3}{1+3k_2};\quad 1+3k_2=k_2-3;\ k_2=-2.$$

The straight line we are seeking passes through the origin; hence its equation is of the form

$$y = kx.$$

Substituting the obtained value of k_2 for k, we get the required equation:

$$y = -2x.$$

3. Given the points $A(-7, -1)$, $B(2, -3)$ and $C(4, -1)$, determine the angle between BC and AB (Fig. 26).

Solution. In accordance with formula (VI) the slope of BC is

$$k_1 = \frac{-1+3}{4-2} = 1,$$

and the slope of AB is

$$k_2 = \frac{-3+1}{2+7} = -\frac{2}{9}.$$

By formula (XXV) we have

$$\tan \Theta = \frac{-\frac{2}{9} - 1}{1 + 1 \cdot \left(-\frac{2}{9}\right)} = -\frac{11}{7} = -1.5714,$$

$$\Theta = 180° - 57°32' = 122°28'.$$

CHAPTER III

QUADRIC CURVES

Sec. 17. Equations of the Circle

1°. Definition. *A circle is the locus of points, in a plane, equidistant (this distance is the radius) from a given point (centre).*

2°. The length of the radius defines the size of the circle and the location of the centre of the circle defines the location of the

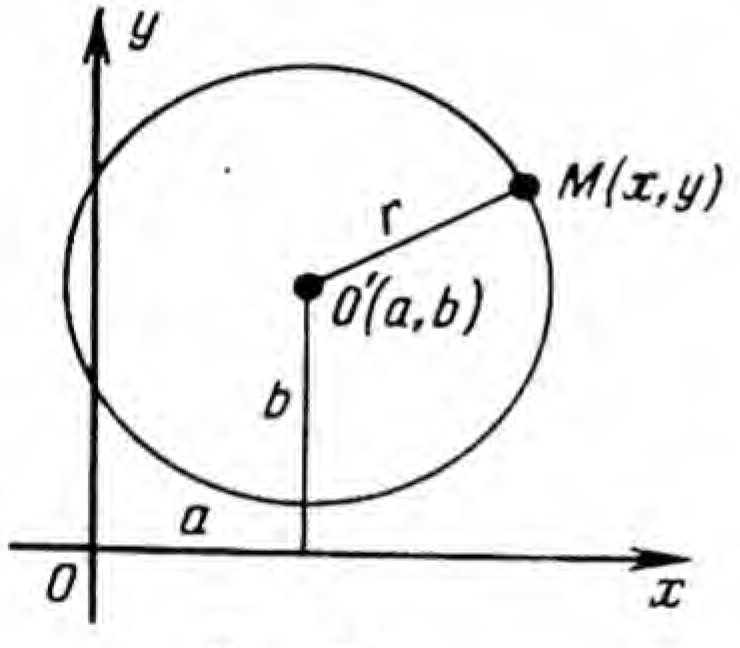

Fig. 27.

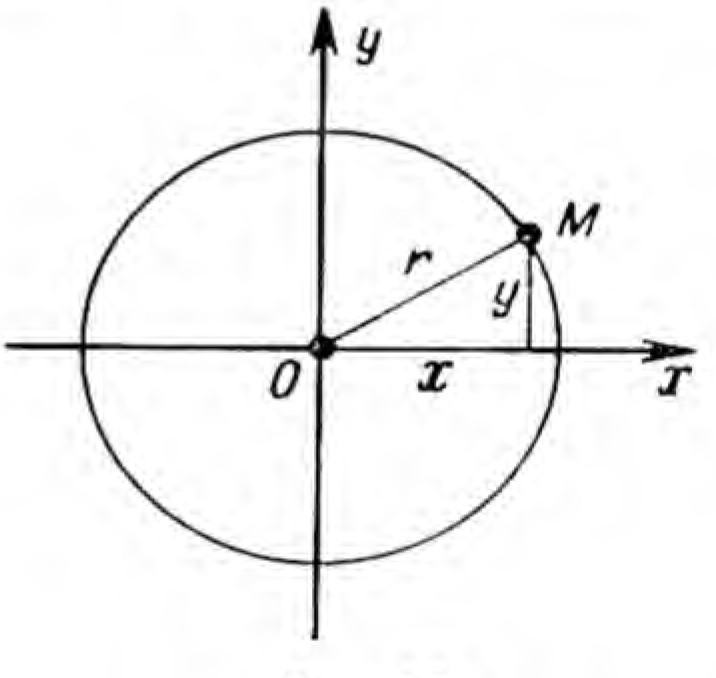

Fig. 28.

circle itself in the plane. Thus a circle is defined as to size and its position in the plane relative to a given coordinate system if we know

1) the length of its radius r and
2) the coordinates a and b of its centre O'.

By definition, any point $M(x, y)$ of a circle (Fig. 27) is at a distance $O'M$, equal to r, from the centre $O'(a, b)$:

$$\overline{O'M} = r.$$

Expressing the distance $O'M$ (by formula II, Sec. 3) in terms of the coordinates of points O' and M, we obtain the equation

$$\sqrt{(x-a)^2 + (y-b)^2} = r.$$

Eliminating the radical by squaring both sides of the equation, we obtain the normal form of the equation of the circle:

$$\boxed{(x-a)^2+(y-b)^2=r^2} \qquad \text{(XXVI)}$$

3°. In particular, when the centre of a circle O' coincides with the origin (Fig. 28),

$$a=0 \text{ and } b=0,$$

the equation of the circle (XXVI) takes the form

$$\boxed{x^2+y^2=r^2} \qquad \text{(XXVII)}$$

Sec. 18. Solved Examples

1°. The values of the parameters a, b and r being given, it is sufficient to introduce them into equation (XXVI) in order to obtain the equation of the circle.

For instance, the equation of a circle with radius equal to 5 and centre in the point $O'(2, -3)$ will be written as

$$(x-2)^2+(y+3)^2-25,$$

since it is given that $a=2$, $b=-3$ and $r=5$.

2°. Write the equation of a circle, the end points of one of the diameters of which are $A(3, -2)$, $B(-4, 5)$ (Fig. 29).

Solution. The centre O' of the circle is the middle point of the segment AB. The coordinates of the middle of AB are (by formula V, Sec. 4):

$$a=\frac{3-4}{2}=-\frac{1}{2}; \quad b=\frac{-2+5}{2}=\frac{3}{2}.$$

The distance AB is the diameter of the circle. From formula II, Sec. 3, we have

$$(2r)^2=AB^2=(3+4)^2+(-2-5)^2=98.$$

Hence

$$r^2=\frac{98}{4}.$$

Introducing the values of a, b and r^2 into formula (XXVI), we obtain the equation of the circle:

$$\left(x+\frac{1}{2}\right)^2+\left(y-\frac{3}{2}\right)^2=\frac{98}{4}.$$

Removing the brackets and transposing $\frac{98}{4}$ to the left-hand side of the equation, we have

$$x^2+x+\frac{1}{4}+y^2-3y+\frac{9}{4}-\frac{98}{4}=0,$$

$$x^2+y^2+x-3y-22=0.$$

3°. Find the equation of a circle which is tangent to the coordinate axes and passes through the point $(-2, 1)$.

Solution. The point $(-2, 1)$ through which the circle passes is in the second quadrant and since the circle is tangent to the coordinate axes it is wholly located in the second quadrant; the abscissa a of its centre is negative and the ordinate b is positive.

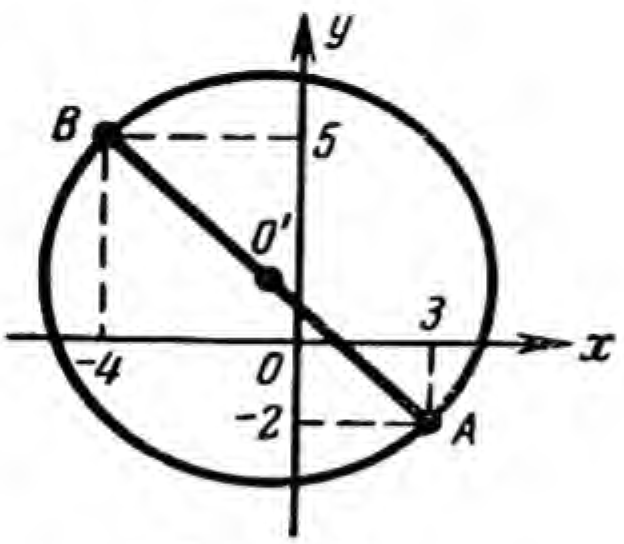

Fig. 29

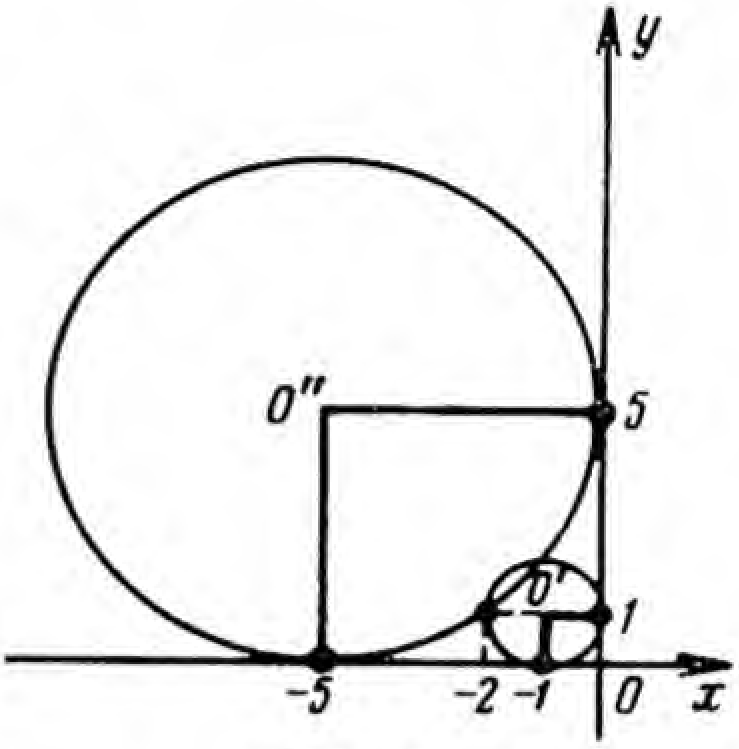

Fig. 30.

The radii drawn to the points of tangency (Fig. 30) are perpendicular to the tangents (i.e., to the axes Ox and Oy). Therefore,

$$a = -r, \; b = r, \; (r > 0).$$

Introducing these values into the equation of the circle (XXVI), we get

$$(x + r)^2 + (y - r)^2 = r^2.$$

We define the value of r for the condition that the circle passes through the point $(-2, 1)$ and its coordinates satisfy the equation written above. Substituting for the moving coordinates x and y their values -2 and 1, we get

$$(-2 + r)^2 + (1 - r)^2 = r^2,$$

$$r^2 - 6r + 5 = 0,$$

$$r_1 = 1, \; r_2 = 5.$$

Consequently there are two possible circles which pass through the point $(-2, 1)$ and are tangent to the coordinate axes. For the first circle we have

$$a = -1, \; b = 1, \; r = 1;$$

for the second circle,

$$a = -5, \; b = 5, \; r = 5.$$

The equation of the first circle is $(x + 1)^2 + (y - 1)^2 = 1$, or

$$x^2 + y^2 + 2x - 2y + 1 = 0.$$

The equation of the second circle is

$$(x + 5)^2 + (y - 5)^2 = 25, \text{ or } x^2 + y^2 + 10x - 10y + 25 = 0.$$

4°. Find the coordinates of the centre and the radius of the following circle:

$$3x^2 + 3y^2 - 4x + 6y - 12 = 0.$$

Solution. Reduce the given equation to the form

$$(x-a)^2+(y-b)^2=r^2. \tag{1}$$

To do this divide the given equation by the coefficient 3 (in the terms x^2 and y^2):

$$x^2+y^2-\frac{4}{3}x+2y-4=0.$$

Combine the terms containing x into one group and those containing y into another. Then we have

$$\left(x^2-\frac{4}{3}x\right)+(y^2+2y)=4$$

or

$$\left(x^2-2\cdot x\cdot\frac{2}{3}\right)+(y^2+2\cdot y\cdot 1)=4.$$

Now complete the first and the second binomials to perfect squares. For this purpose we add $\frac{4}{9}=\left(\frac{2}{3}\right)^2$ to the first binomial and $1=1^2$ to the second one, and so as to preserve the equal sign in the equation we add $\frac{4}{9}$ and 1 to the right-hand side of the equation as well. Thus we have

$$\left(x^2-2\cdot x.\frac{2}{3}+\frac{4}{9}\right)+(y^2+2\cdot y\cdot 1+1)=4+\frac{4}{9}+1$$

or

$$\left(x-\frac{2}{3}\right)^2+(y+1)^2=\frac{49}{9}.$$

Comparing this equation with (1) we conclude that

$$a=\frac{2}{3},\quad b=-1,\quad r=\frac{7}{3}=2\frac{1}{3}.$$

Sec. 19. The Circle as a Quadric Curve

1° The degree of the equation of a curve, after it has been reduced to integral and rational form with respect to the coordinates x and y, is called *the degree of the curve.*

A straight line is a curve of the first degree since it is expressed by an equation of the first degree. The circle is a curve of the second degree (quadric curve). Indeed, if we remove the brackets in the equation of the circle

$$(x-a)^2+(y-b)^2=r^2$$

and arrange the terms in descending order of powers of the current coordinates x and y, we obtain:

$$x^2+y^2-2ax-2by+(a^2+b^2-r^2)=0. \tag{1}$$

The parameters a, b, r (all of them or some of them) may happen to be fractions. Then, multiplying (1) by their least common multiple A, we obtain

$$Ax^2+Ay^2-2aAx-2bAy+A(a^2+b^2-r^2)=0. \tag{2}$$

Putting $-2aA = D$, $-2bA = E$ and $A(a^2+b^2-r^2) = F$, we obtain the general form of the equation of the circle

$$\boxed{Ax^2 + Ay^2 + Dx + Ey + F = 0} \qquad \text{(XXVIII)}$$

This is an equation of the second degree between the coordinates x and y and, therefore, the circle is a quadric curve.

2°. The general form of an equation of the second degree in two variables is, as we know from algebra, as follows:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \qquad (3)$$

An equation of the second degree between the coordinates x, y, in which the coefficients of the squares of x and y are equal and there is no term with the product xy, represents a circle.

This is a necessary condition. Indeed, in the general form of the equation of the circle (XXVIII) the coefficients of the squares of x and y are equal and there is no term with the product xy.

This condition is sufficient. Indeed, dividing the equation $Ax^2 + Ay^2 + Dx + Ey + F = 0$ by A

$$x^2 + y^2 + \frac{D}{A}x + \frac{E}{A}y + \frac{F}{A} = 0$$

and writing the resulting quotient in the form

$$\left(x^2 + \frac{D}{A}x\right) + \left(y^2 + \frac{E}{A}y\right) = -\frac{F}{A},$$

we complete the squares in $x^2 + \frac{D}{A}x$ and $y^2 + \frac{E}{A}y$:

$$\left(x^2 + 2\frac{D}{2A}x + \frac{D^2}{4A^2}\right) + \left(y^2 + 2\cdot\frac{E}{2A}y + \frac{E^2}{4A^2}\right) = \frac{D^2}{4A^2} + \frac{E^2}{4A^2} - \frac{F}{A}$$

and obtain

$$\left(x + \frac{D}{2A}\right)^2 + \left(y + \frac{E}{2A}\right)^2 = \frac{D^2 + E^2 - 4AF}{4A^2}.$$

That is, we obtain the equation of a circle for which

$$\boxed{a = -\frac{D}{2A}, \quad b = -\frac{E}{2A}, \quad r^2 = \frac{D^2 + E^2 - 4AF}{4A^2}} \qquad \text{(XXIX)}$$

3°. We shall now point out some special cases which may occur for certain values of the coefficients A, D, E and F.

Example. The equation

$$x^2 + y^2 - 6x + 4y + 13 = 0$$

may be given in the form

$$x^2-6x+9+y^2+4y+4=0$$

or

$$(x-3)^2+(y+2)^2=0.$$

Since none of the squares $(x-3)^2$ or $(y+2)^2$ can be negative, their sum can equal zero only if

$$x-3=0 \text{ and } y+2=0.$$

Whence

$$x=3 \text{ and } y=-2.$$

This equation is satisfied by only one point in the plane, namely $(3, -2)$, and represents a circle of zero radius, $r=0$.

If in the same equation we take, say, 15 instead of 13 as the absolute term, that is, the equation

$$x^2+y^2-6x+4y+15=0,$$

then after reducing it to the normal form we get

$$(x-3)^2+(y+2)^2=-2.$$

This equation is not satisfied by any pair of real values of the current coordinates, i.e., by any point in the plane. However, for the sake of generality, it is considered that in this case also the equation represents a circle, an *imaginary* circle, with radius $r=\sqrt{-2}$.

Sec. 20. Ellipse

1°. **Definition.** *An ellipse is the locus of points in a plane for each of which the sum of its distances from two given points (foci) is constant.*

By definition, if F and F_1 (Fig. 31) are the two given points in the plane, called the foci of the ellipse, and M is an arbitrary point of the ellipse, then the sum of the distances $\overline{MF_1}$ and $\overline{MF}$ is constant and is taken equal to $2a$, i.e.,

$$\overline{MF_1}+\overline{MF}=2a, \quad (a>0).$$

$\overline{MF_1}$ and $\overline{MF}$ are called the radii vectors of the point M.

$\overline{F_1F}$ is called the focal length and is taken equal to $2c$,

$$\overline{F_1F}=2c.$$

Considering the triangle F_1FM, we see that $\overline{MF_1}+\overline{MF}>\overline{F_1F}$, i.e., $2a>2c$, or $a>c$, $(c>0)$.

2°. Take a piece of thread $2a$ in length and fix its ends at points F and F_1 with two pins at a separation of $2c$. Then make the thread taut with the point of a pencil as shown in Fig. 32

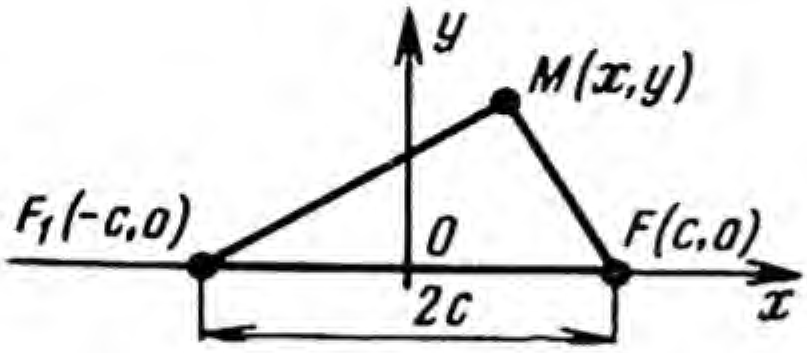

Fig. 31.

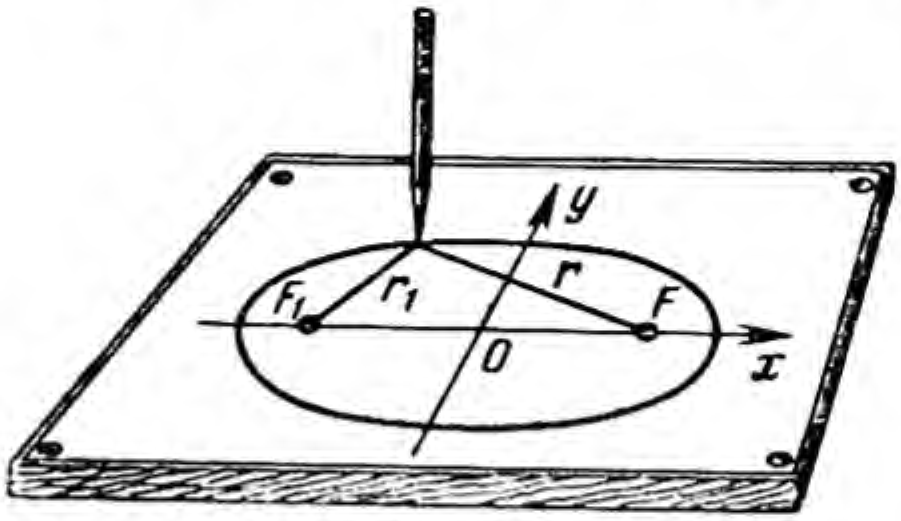

Fig. 32.

and describe a curve with it keeping the thread taut all the time. In this manner an ellipse can be drawn in two steps: first on one side of the straight line F_1F and then on the other.

Sec. 21. The Equation of an Ellipse

Let us take the middle of the focal length as the coordinate origin O, the straight line F_1F as the axis Ox and the perpendicular to it passing through the point O as the axis Oy (Fig. 31).

The foci F_1 and F then have coordinates $(-c, 0)$ and $(+c, 0)$.

Inasmuch as the position of point M can vary with respect to the coordinate axes, its coordinates will be the current coordinates (x, y).

$$\overline{MF_1} = \sqrt{(x+c)^2+y^2}, \quad \overline{MF} = \sqrt{(x-c)^2+y^2}. \qquad (1)$$

By the definition of an ellipse,

$$\overline{MF_1} + \overline{MF} = 2a.$$

Therefore

$$\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} + 2a.$$

Let us get rid of the radicals. To do this, first transpose one of the radicals to the right-hand side,

$$\sqrt{(x+c)^2+y^2} = 2a - \sqrt{(x-c)^2+y^2},$$

and then square both sides of the equation.

Removing the brackets we have

$$x^2+2cx+c^2+y^2 = 4a^2 -$$
$$- 4a\sqrt{(x-c)^2+y^2} + x^2 - 2cx + c^2 + y^2.$$

Eliminating x^2, c^2, y^2 and transposing the radical to the left side and $2cx$ to the right side of the equality sign, we obtain

$$4a\sqrt{(x-c)^2+y^2}=4a^2-4cx,$$

or, dividing the equation by $4a$,

$$\sqrt{(x-c)^2+y^2}=a-\frac{c}{a}x. \tag{2}$$

By squaring both sides of the equation and removing the brackets we have

$$x^2-2cx+c^2+y^2=a^2-2cx+\frac{c^2}{a^2}x^2.$$

Eliminating $-2cx$ and transposing $\frac{c^2}{a^2}x^2$ to the left side and c^2 to the right side of the equality sign, we get

$$x^2-\frac{c^2}{a^2}x^2+y^2=a^2-c^2.$$

Multiplying by a^2 we then have

$$(a^2-{}^2c)\,x^2+a^2y^2=a^2(a^2-c^2).$$

Since $a>c$ we may put

$$\boxed{a^2-c^2=b^2} \tag{XXX}$$

and write the preceding equation in the form

$$b^2x^2+a^2y^2=a^2b^2.$$

We divide this equation by a^2b^2 and obtain the canonical equation of the ellipse:

$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1} \tag{XXXI}$$

Sec. 22. Investigating the Form of an Ellipse from Its Equation

1°. Let us solve the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for y,

$$y=\pm\frac{b}{a}\sqrt{a^2-x^2},$$

and investigate y as a function of x. For the values of y to be real numbers, the radicand a^2-x^2 must be either positive or equal to zero. This will be the case if

$$|x|\leqslant a.$$

Consequently x can take on values only in the interval

$$-a \leqslant x \leqslant +a,$$

where $a > 0$ (Fig. 33).

If $x = \pm a$, then $a^2 - x^2 = 0$ and $y = 0$.

The points of intersection of the ellipse with the axis of abscissas, $A(a, 0)$ and $A_1(-a, 0)$, are called the *vertices* of the ellipse

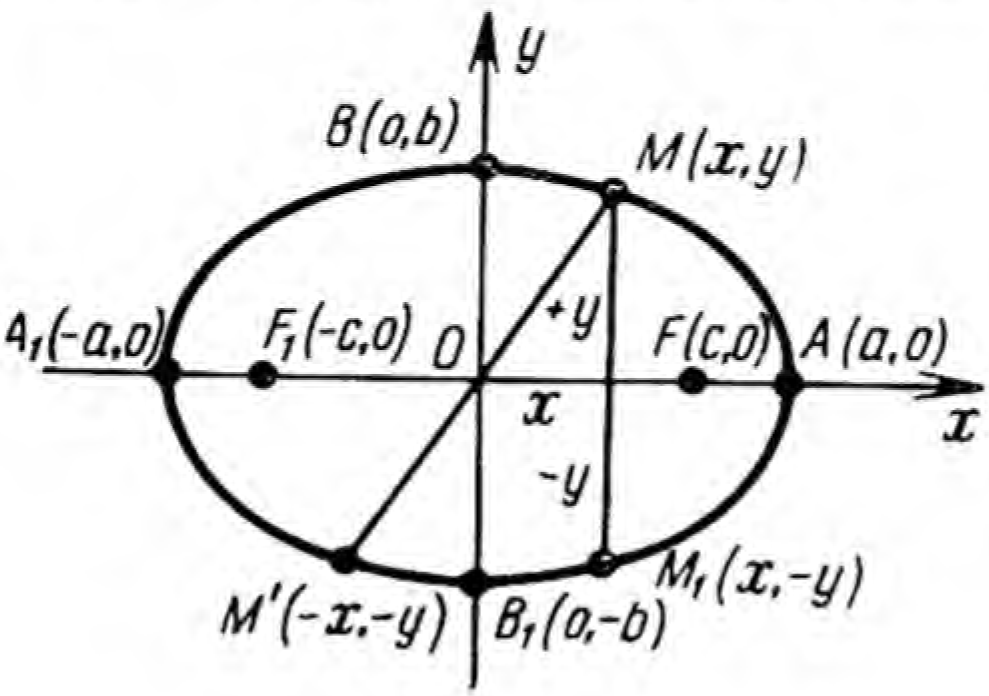

Fig. 33.

and the chord $A_1A = 2a$ is called the *major axis* of the ellipse. If $x = 0$, then $y = \pm \frac{b}{a}\sqrt{a^2} = \pm b$. The points $B(0, b)$ and $B_1(0, -b)$ are the points of intersection of the ellipse with the axis of ordinates and are also called the *vertices* of the ellipse. The chord $B_1B = 2b$ is called the *minor axis* of the ellipse.

To each value of x in the interval $[-a, +a]$ there correspond two points of the ellipse located on different sides of the axis Ox at a distance equal to the absolute value of y, since to each value of x there correspond two values of y equal in absolute value and opposite in sign. Thus, the ellipse is symmetric about the axis Ox.

When x increases from $-a$ to zero, the absolute value of y increases from zero to b; and when x increases from zero to a, the absolute value of y decreases from b to zero.

The equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be solved for x and not y:

$$x = \pm \frac{a}{b}\sqrt{b^2 - y^2}$$

and we can analyse it in the same manner as before. We shall then come to the following conclusions: x has real values only when y varies in the interval $-b \leqslant y \leqslant b$; when y increases from $-b$ to zero, the absolute value of x increases from zero to a; and when y increases from zero to b, the absolute value of x decreases from a to zero. To each value of y in the interval $-b \leqslant y \leqslant b$ there correspond two values of x, equal in absolute

value and opposite in sign. The ellipse is symmetric about the axis Oy.

2°. The current coordinates x, y are contained in the equation of the ellipse only as squares. Therefore, if the point (x, y) belongs to the ellipse, then the point $(-x, -y)$ also belongs to it because $(-x)^2 = x^2$ and $(-y)^2 = y^2$. The chord connecting the points (x, y) and $(-x, -y)$ of the ellipse has the origin O as its middle point, since by formula V, Sec. 4.

$$\frac{x+(-x)}{2}=0 \text{ and } \frac{y+(-y)}{2}=0.$$

The point bisecting all the chords of the curve passing through it is called the centre of the curve. The origin O is the centre of the ellipse.

Sec. 23. Plotting an Ellipse

1°. *Plotting by points* (Fig. 34). Given $2a$ and $2c$. On a straight line lay off a segment $F_1F = 2c$ and halve it. Then, from the centre O thus obtained lay off on the straight line F_1F segments

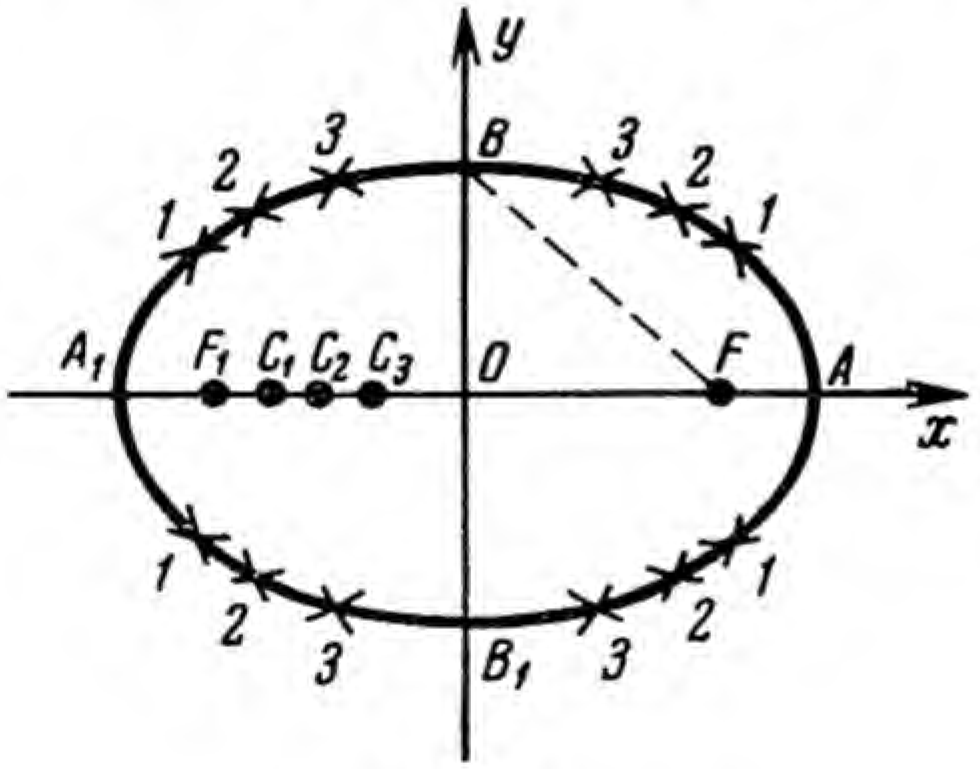

Fig. 34.

OA_1 and OA equal to a to the left and right of O, respectively. We thus obtain the major axis A_1A. Through the centre O draw a straight line B_1B perpendicular to A_1A. From the focus F as the centre describe an arc with radius equal to a. The arc will intersect the perpendicular B_1B at points B_1 and B. The segment $B_1B = 2b$ is the minor axis of the ellipse, since from the triangle OFB it follows that

$$\overline{OB^2} = \overline{FB^2} - \overline{OF^2} = a^2 - c^2 = b^2$$

(formula XXX). Thus we have constructed four points which are the vertices of the ellipse: A_1 A, B_1 and B.

In order to construct other points of the ellipse, let us take on the segment F_1F to the left (or to the right) of the centre an arbitrary point C_1 and from the foci F and F_1 as centres strike arcs (each time one arc above the straight line A_1A and another below it) first with the radius equal to $\overline{A_1C_1}$ and then with the radius equal to $\overline{C_1A}$.

The intersections of these arcs will give four points (in the figure all four are labelled 1).

The points 1 belong to the ellipse because the sum of the distances of each of them from the foci is $2a$:

$$\overline{1F_1}+\overline{1F}=\overline{A_1C_1}+\overline{C_1A}=\overline{A_1A}=2a.$$

Taking on the segment OF_1 (or OF) other points $C_2, C_3, \ldots$ and performing the same operations as in the case of the point C_1, we will each time obtain 4 points of the ellipse (in the figure they are labelled 2, 3, etc.).

Having thus constructed a sufficient number of points, draw by hand or with the aid of a curved ruler a smooth continuous curve — the ellipse. Note that when constructing the ellipse, it is necessary to take more points on the segment F_1O near the focus F_1 than near the centre O, and points $C_1, C_2, C_3, \ldots$ should be taken closer to each other as they approach F_1.

2°. When the semiaxes a and b are given, it is necessary first to find c. To do this it is sufficient to construct a right-angled triangle with the hypotenuse equal to the semimajor axis a, one side equal to the semiminor axis b, and the other side equal to c (formula XXX).

Sec. 24. Relationship Between the Ellipse and the Circle

1°. If in the equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ we put $b=a$, we get

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}=1 \text{ or } x^2+y^2=a^2,$$

that is, the equation of a circle of radius a. Thus a circle is an ellipse with equal semiaxes.

2°. Let us describe (Fig. 35) from the centre O of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ a circle with radius equal to the semimajor axis a; its equation is $x^2+y^2=a^2$. Let us solve the equations of the ellipse and the circle for y and, in order to distinguish the ordinates of the points of the ellipse from those of the circle having the same abscissa x, let us designate the ordinate of a point of

the ellipse by y_e, and that of the circle by y_c. Then we have

$$y_e = \pm \frac{b}{a}\sqrt{a^2 - x^2},$$

$$y_c = \pm \sqrt{a^2 - x^2}.$$

Dividing the first equation by the second, we obtain

$$\frac{y_e}{y_c} = \frac{b}{a},$$

i.e., *if a circle is drawn with the major axis of an ellipse as its diameter, then for each value of the abscissa x the ratio between*

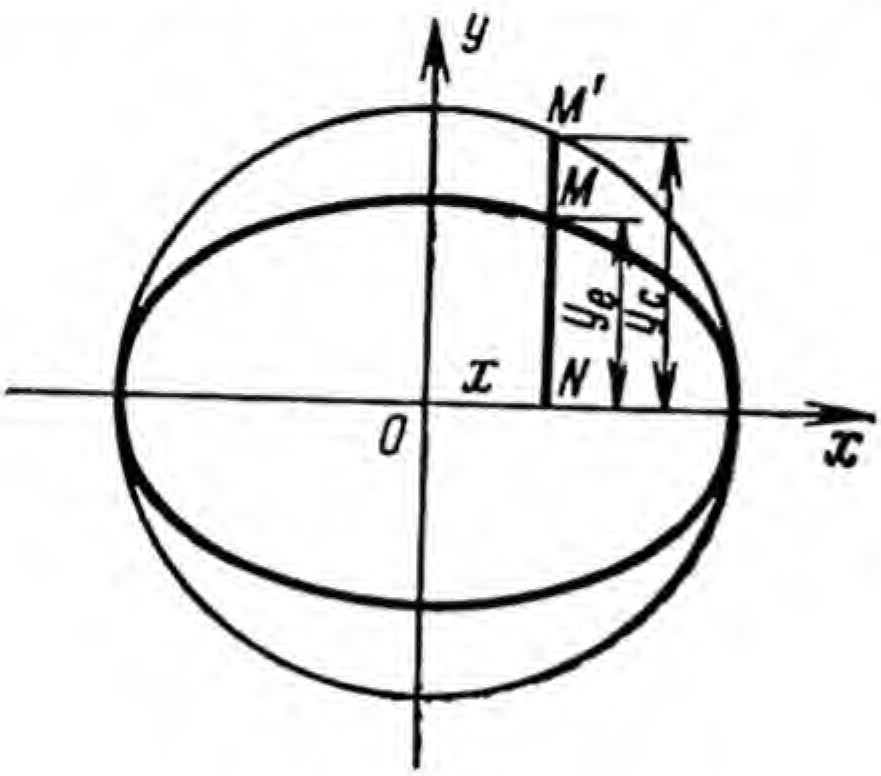

Fig. 35.

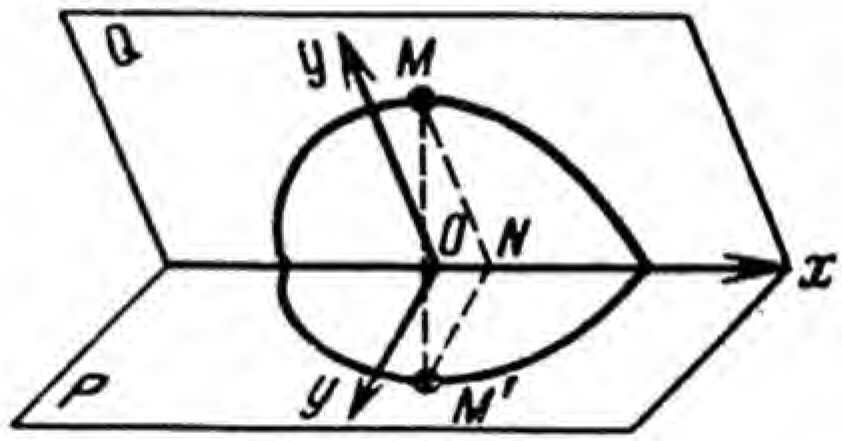

Fig. 36.

the ordinates of the corresponding points of the ellipse and those of the circle is constant and equal to the ratio of the semiminor axis of the ellipse to its semimajor axis.

Whence it follows that an ellipse can be obtained by "uniform compression" of a circle of radius a, i.e., by the reduction of all semichords (normal to its diameter) in the constant ratio $\frac{b}{a}$. This ratio $\frac{b}{a}$ is called the coefficient of compression.

3°. Let the plane Q of a circle $x^2+y^2=a^2$ form with plane P an angle φ (Fig. 36). By dropping a perpendicular on the plane P from each point M of the given circle we obtain the orthogonal projection of the circle on the plane P. From Fig. 36 it follows that if $MM' \perp P$ and $MN \perp Ox$, then $\angle M'NM = \varphi$. From the right-angled triangle $M'NM$ we have $\frac{M'N}{MN} = \cos\varphi$, i.e., all semichords of the circle normal to the diameter Ox are represented in the plane P as reduced in one and the same ratio equal to $\cos\varphi$. Therefore *the orthogonal projection of a circle on the plane P is an ellipse.*

Sec. 25. Eccentricity of an Ellipse

1°. The ratio of the focal length to the length of the major axis of an ellipse is called the eccentricity of the ellipse and is denoted by the letter e:

$$\boxed{e = \frac{c}{a}} \qquad \text{(XXXII)}$$

Since $c < a$, $e < 1$.

The semiaxes a and b of the ellipse being given, we can find its eccentricity. From formula (XXX)

$$c^2 = a^2 - b^2, \qquad c = \sqrt{a^2 - b^2}, \qquad (c > 0).$$

Introducing the value of c into formula (XXXII) we obtain

$$\boxed{e = \frac{\sqrt{a^2 - b^2}}{a}} \qquad \text{(XXXIII)}$$

From this formula it follows that if $b = a$, i.e., if the ellipse is a circle, the eccentricity $e = 0$; if a remains unchanged and b decreases from a to zero, the eccentricity of the ellipse increases from zero to unity; the eccentricity is equal to unity when the ellipse turns into the segment A_1A of a straight line.

2°. In the derivation of the equation of the ellipse (Sec. 21) we obtained equality (2):

$$\sqrt{(x-c)^2 + y^2} = a - \frac{c}{a}x.$$

In it, $\sqrt{(x-c)^2 + y^2} = \overline{MF}$ [Eq. (1), Sec. 21]—the radius vector of the point M. Let us denote it by r. The ratio $\frac{c}{a} = e$. Therefore

$$\boxed{r = a - e \cdot x}$$

Denoting the second radius vector of the point M by r_1, $\overline{MF_1} = r_1$, we obtain

$$r_1 = a + ex,$$

as

$$r + r_1 = 2a.$$

Sec. 26. Hyperbola

Definition. *A hyperbola is the locus of points in a plane, for each of which the difference of its distances from two given points (foci) is constant.*

In accordance with the definition, if F_1 and F (Fig. 37) are the given points in the plane called the foci of the hyperbola, and M is an arbitrary point of the hyperbola, then the difference between the distances $\overline{MF_1}$ and $\overline{MF}$ (where $\overline{MF_1} > \overline{MF}$, or, possibly, $\overline{MF_1} < \overline{MF}$) is constant and is taken equal to $2a$:

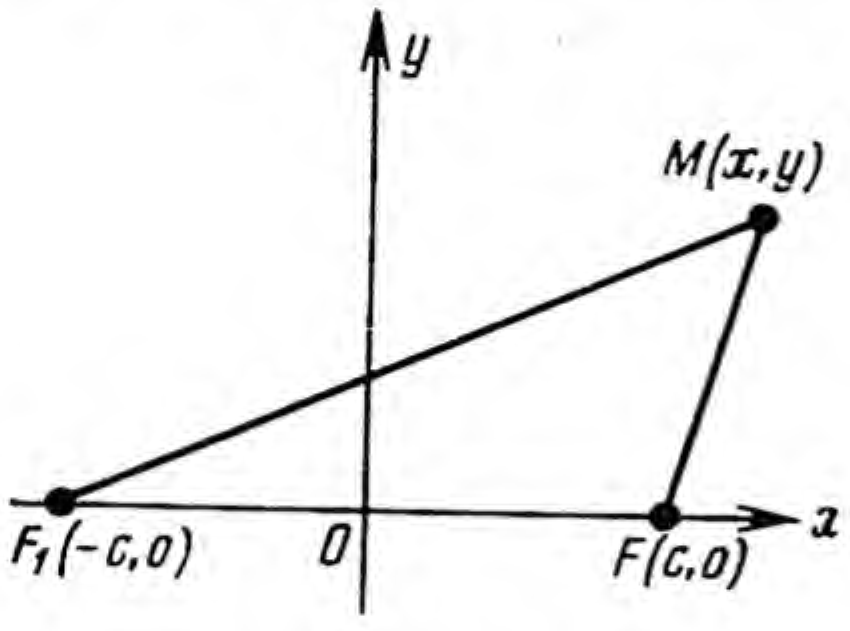

Fig. 37.

$$|\overline{MF_1} - \overline{MF}| = 2a. \quad (a > 0).$$

$\overline{MF_1}$ and $\overline{MF}$ are called the radii vectors of the point M.

F_1F is called the focal length and is taken equal to $2c$:

$$\overline{F_1F} = 2c.$$

It is evident from the triangle F_1MF that

$$|\overline{MF_1} - \overline{MF}| < \overline{F_1F}, \quad \text{i.e.,} \quad 2a < 2c \quad \text{or } a < c.$$

Sec. 27. The Equation of the Hyperbola

Let us take the middle point of the focal length for the origin O (Fig. 37), the straight line F_1F for the axis Ox and a perpendicular to it passing through the point O for the axis Oy. The foci F_1 and F have the coordinates $(-c, 0)$ and $(+c, 0)$. The point M has the current coordinates (x, y).

$$\begin{aligned} \overline{MF_1} &= \sqrt{(x+c)^2+y^2}, \\ \overline{MF} &= \sqrt{(x-c)^2+y^2}. \end{aligned} \tag{1}$$

In accordance with the definition of the hyperbola,

$$|\overline{MF_1} - \overline{MF}| = 2a,$$

therefore $\left|\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2}\right| = 2a.$

Whence

$$\sqrt{(x+c)^2+y^2} = \sqrt{(x-c)^2+y^2} = \pm 2a.$$

By squaring both sides of the equation and simplifying it we obtain

$$4cx - 4a^2 = \pm 4a\sqrt{(x-c)^2+y^2}.$$

Dividing by $4a$ termwise we get

$$\frac{c}{a}x - a = \pm\sqrt{(x-c)^2+y^2}. \tag{2}$$

After squaring both sides of equation (2) and simplifying, we obtain

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2).$$

Since $c > a$, we may put

$$\boxed{c^2 - a^2 = b^2} \tag{XXXIV}$$

The preceding equation may be written in the form

$$b^2x^2 - a^2y^2 = a^2b^2.$$

We divide this equation by a^2b^2 and obtain the canonical (standard) equation of the hyperbola:

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1} \tag{XXXV}$$

Sec. 28. Investigating the Forms of the Hyperbola from Its Equation

1°. Let us solve the equation of the hyperbola (XXXV) for y,

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2}\,. \tag{1}$$

The values of y are real numbers if $x^2 - a^2 \geqslant 0$ or if

$$|x| \geqslant a.$$

If $-a \leqslant x \leqslant +a$, then to the values of x there correspond imaginary values of y, and, therefore, if we draw straight lines $x = -a$ and $x = a$ (Fig. 38), there will be no points of the hyperbola in the infinite region enclosed between these lines. Thus the hyperbola does not intersect the axis Oy.

If $x = \pm a$, then $x^2 - a^2 = 0$ and $y = 0$. The points $A(a, 0)$ and $A_1(-a, 0)$ are the points of intersection of the hyperbola with the axis of abscissas and are called the *vertices* of the hyperbola. The chord $A_1A = 2a$ is called the *real axis* of the hyperbola.

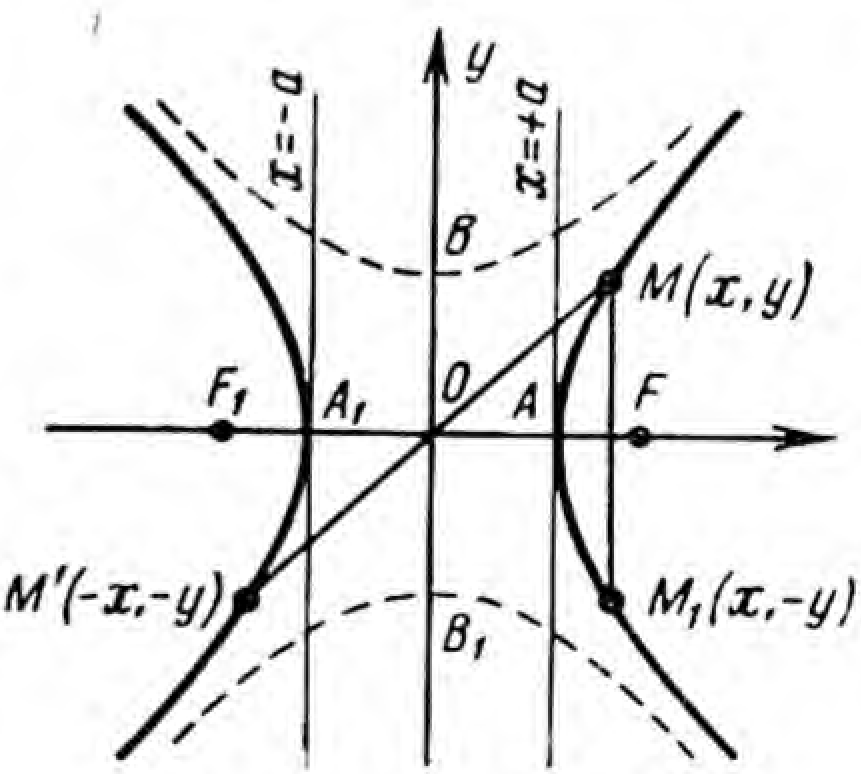

Fig. 38

To each value of x in the intervals $-\infty < x < -a$ and $+a < x < +\infty$ there correspond two points of the hyperbola located on both sides of the axis Ox and at a distance from the axis equal to the absolute value of y, since to each value of x in these intervals there correspond two values of y equal in

magnitude and opposite in sign. The hyperbola is a curve symmetric about the axis Ox.

When x increases from a to $+\infty$, the absolute value of y increases from zero to infinity. When x decreases from $-a$ to $-\infty$, the adsolute value of y also increases, taking successively the same values as in the case of the increase of x from a to $+\infty$, because x is contained in equation (1) only in the second power and, therefore,

$$\sqrt{(-x)^2-a^2}=\sqrt{(+x)^2-a^2}, \quad (|x|\geqslant a).$$

Thus, a hyperbola consists of two branches of the same form, symmetric about the axes Ox and Oy, one of which is located to the right of the straight line $x=a$ and the other to the left of the straight line $x=-a$, both branches extending to infinity.

On the axis Oy upwards and downwards from the origin O we lay off a segment of length b. From formula (XXXIV) it is evident that this segment is a leg of a right-angled triangle in which the hypotenuse is equal to c and the other leg is equal to a: The segment $B_1B=2b$ is called the *imaginary axis* of the hyperbola. The points B_1 and B are called the *imaginary vertices* of the hyperbola.

2°. Since in the equation of the hyperbola (XXXV) the current coordinates x and y are contained only in the second power, if a point (x, y) belongs to the hyperbola, then the point $(-x, -y)$, symmetric about the origin O, must also belong to it. Thus, *the origin O is the centre of the hyperbola.*

3°. If we take the segment $B_1B=2b$ for the real axis and the segment $A_1A=2a$ for the imaginary axis, the equation of the hyperbola will take the form

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

or

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1,$$

and the hyperbola itself will take the form represented in Fig. 38 by the dashed curves.

Two hyperbolas, the equations of which are

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \text{and} \quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=-1,$$

are called *conjugate*.

Sec. 29. Plotting a Hyperbola

1°. *By continuous motion* (Fig. 39). Given $2a$ and $2c$. Take a ruler (or rod) and fix a thread to one end. The other end of the ruler and thread fix loosely (with pins, for example) in the foci F_1

and F. The length of the thread must be such that the difference between the length of the ruler (F_1N) and that of the thread (FMN) is equal to $2a$. With the point of a pencil make the thread taut so that the ruler coincides with the straight line F_1F; then the point of the pencil will be in the vertex A of the hyperbola. Then draw a curve, keeping the thread taut all the time

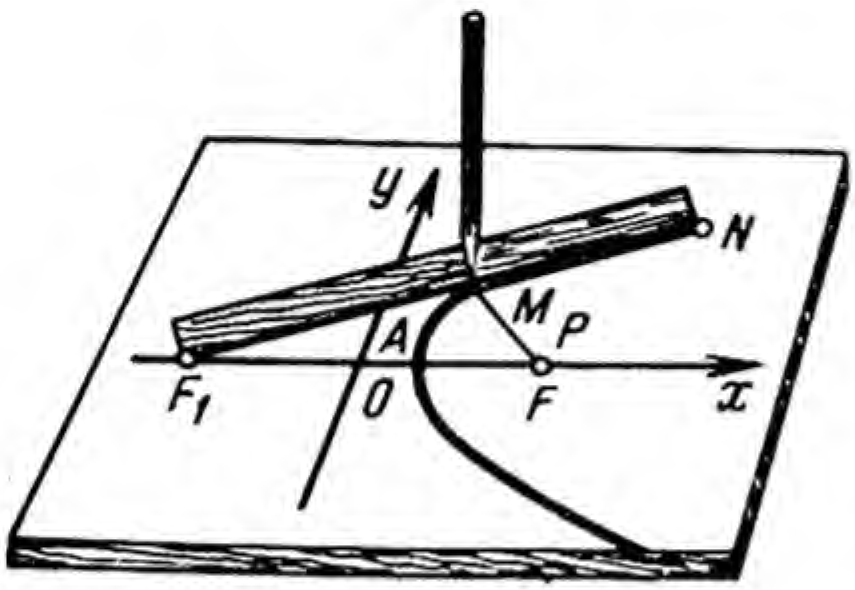

Fig. 39.

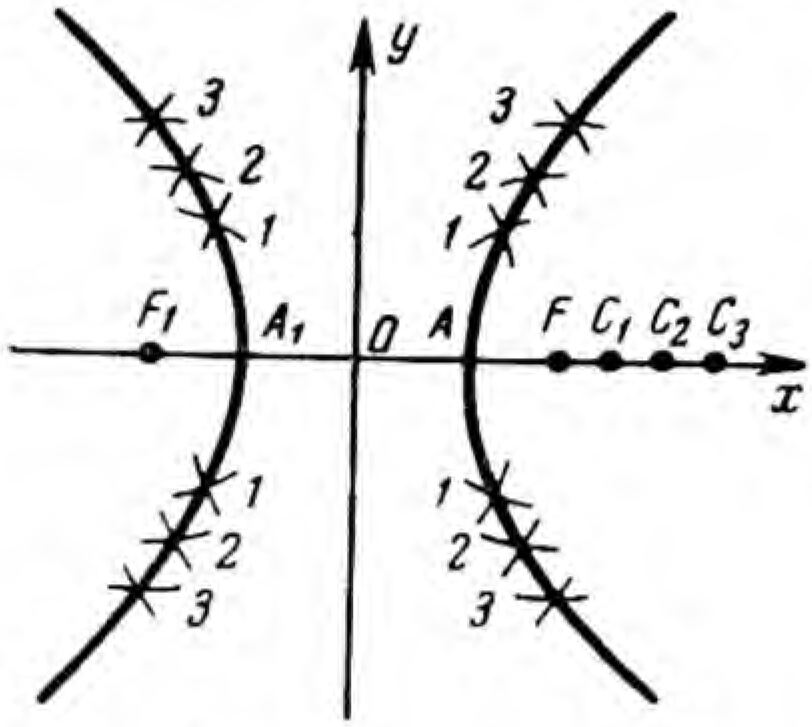

Fig. 40.

the pencil moves on the paper. In this manner one branch of the hyperbola may be drawn in two steps: first on one side of the straight line F_1F and then on the other.

To draw the second branch of the hyperbola, transfer the centre of rotation of the ruler from focus F_1 to focus F and fix the free end of the thread in focus F_1.

2°. *By points* (Fig. 40). Given $2a$ and $2c$. On a straight line lay off a segment $F_1F=2c$ and then halve it. From the centre O thus obtained lay off on F_1F segments OA_1 and OA, to the left and right of O, each equal to a. They will form the real axis A_1A of the hyperbola.

On the extension of the real axis take a number of points C_1, C_2, C_3, ... to the right of the focus F (or to the left of the focus F_1). With the foci F and F_1 as centres, strike arcs (each time one above the straight line A_1A and one below it) first with radius equal to $\overline{A_1C_1}$, then with radius equal to $\overline{C_1A}$. The intersections of the arcs will give four points of the hyperbola (labelled *1* in the figure).

Points *1* belong to the hyperbola because the difference between the radii vectors of each of them is $2a$:

$$|\overline{1F_1}-\overline{1F}|=|\overline{A_1C_1}-\overline{C_1A}|=\overline{A_1A}=2a.$$

Performing similar operations for the points C_2, C_3, ... each time we get four new points of the hyperbola (labelled *2*, *3*, ... in the figure).

Having constructed a sufficient number of points of the hyperbola in the manner outlined above, we draw a smooth curve through them either by hand or with the help of a curved ruler.

3°. When the semiaxes a and b are given, it is first necessary to find c. To do this it is sufficient to construct a right-angled triangle, the legs of which are equal to the semiaxes a and b. The hypotenuse of this triangle will be c, since from formula (XXXIV) it follows that $a^2+b^2=c^2$.

Sec. 30. Asymptotes of the Hyperbola

Let us examine the relative positions of the straight line (Fig. 41)

$$y=\frac{b}{a}x \tag{1}$$

and the right branch of the hyperbola in the first quadrant. We represent the equation of the hyperbola (XXXV) in the form:

$$y=+\frac{b}{a}\sqrt{x^2-a^2}. \tag{2}$$

We will regard x as an arbitrary abscissa in the interval $a<x<+\infty$. Substituting this value of x into the equation of the straight line (1) and the equation of a branch of the hyperbola (2), we find the ordinate of the point whose abscissa is x, which point lies on the straight line (1) and the hyperbola (2). Let us denote their values by y_a and y_h, i.e.,

$$y_a=\frac{b}{a}x, \quad y_h=\frac{b}{a}\sqrt{x^2-a^2}.$$

Since $a^2>0$, $x>\sqrt{x^2-a^2}$ and $y_a>y_h$. The difference between the ordinates is

$$y_a-y_h=\frac{b}{a}x-\frac{b}{a}\sqrt{x^2-a^2}$$

or

$$y_a-y_h=\frac{b\,(x-\sqrt{x^2-a^2})}{a}.$$

Rationalising the numerator of the fraction, we obtain

$$y_a-y_h=\frac{b\,(x-\sqrt{x^2-a^2})\,(x+\sqrt{x^2-a^2})}{a\,(x+\sqrt{x^2+a^2})}=$$

$$=\frac{b\,(x^2-x^2+a^2)}{a\,(x+\sqrt{x^2-a^2})}=\frac{ab}{x+\sqrt{x^2-a^2}}.$$

The numerator ab of this fraction is a constant, while the magnitude of the denominator $x+\sqrt{x^2-a^2}$ depends on x and increases with x. Consequently, the difference y_a-y_h decreases with increase in x. When x increases indefinitely, y_a-y_h tends to zero.

Let point P on a straight line and point M on the hyperbola correspond to some value of x. Now $y_a - y_h = \overline{MP}$. Drop a perpendicular MQ on the straight line OP from point M. In the right-angled triangle MQP $\angle PMQ = \angle xOP = \varphi$, $\overline{MQ} = \overline{MP} \cdot \cos\varphi$.

Since $\cos\varphi = \text{const}$ and $\overline{MP} = y_a - y_h$ tends to zero as x increases indefinitely, $\overline{MQ}$ must also tend to zero as x increases indefinitely.

A straight line with the property that a perpendicular dropped on it from a point on the curve tends to zero when the point recedes

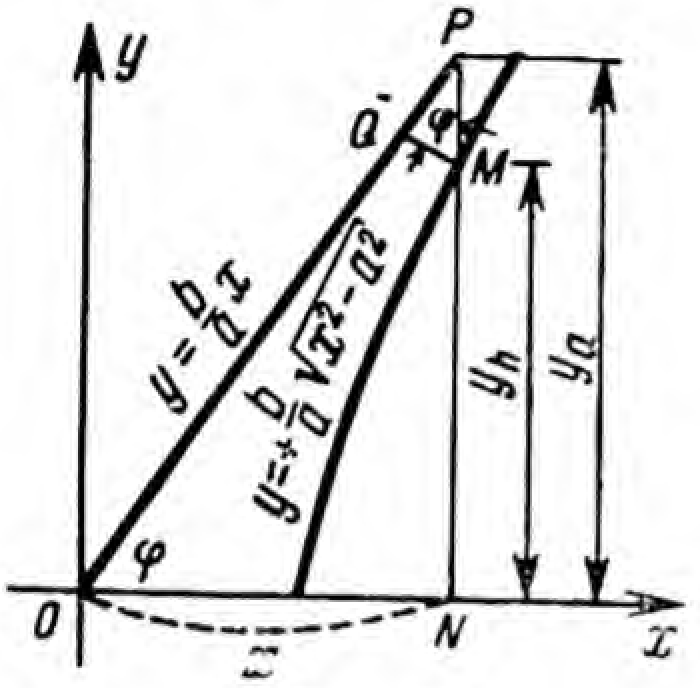

Fig. 41.

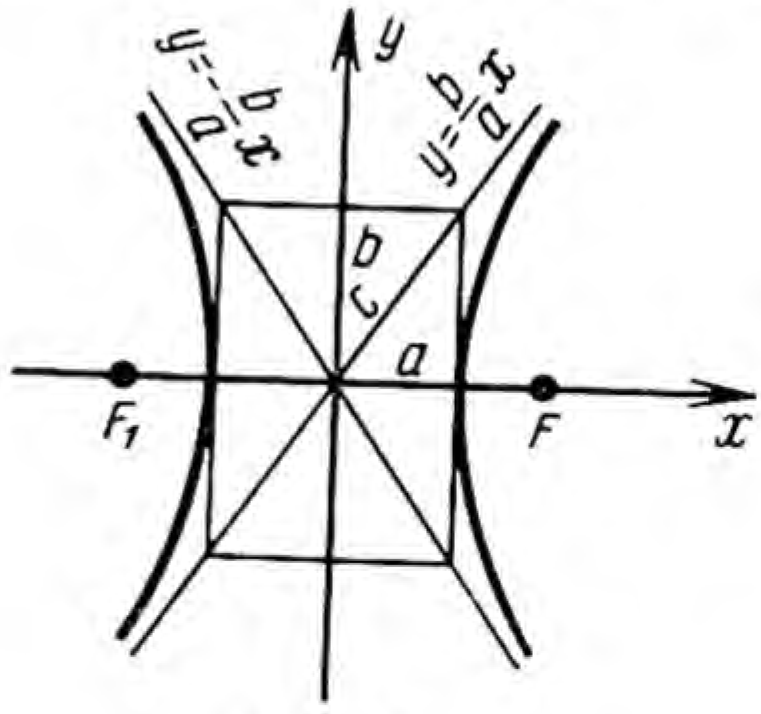

Fig. 42.

indefinitely along an infinite branch of the curve is called an *asymptote of the curve*. Thus, the straight line $y = \frac{b}{a}x$ is an asymptote of the right branch of the hyperbola. It is also asymptotic to the left branch of the curve.

Similarly the straight line $y = -\frac{b}{a}x$ is an asymptote of both branches of the curve. This is a consequence of the symmetry of the hyperbola with respect to the coordinate axes.

Thus, a hyperbola has two asymptotes:

$$\boxed{y = \pm \frac{b}{a} x} \qquad \text{(XXXVI)}$$

To construct the asymptotes of a hyperbola it is sufficient to construct a rectangle whose axes of symmetry are the axes of the hyperbola (Fig. 42) and then draw the diagonals of this rectangle. Since the equations of these diagonals are $y = \pm \frac{b}{a}x$, they yield the asymptotes of the curve when produced indefinitely.

The right and left branches of the hyperbola lie entirely inside the angles formed by the two asymptotes $y = +\frac{b}{a}x$ and $y = -\frac{b}{a}x$ at their intersection. On extension to infinity, the branches approach the asymptotes.

The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$, which is conjugate to the given hyperbola, viz., $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, has the same asymptotes as the latter.

Sec. 31. Eccentricity of a Hyperbola

1°. The ratio of the focal length $2c$ to the length of the real axis $2a$ of a hyperbola is called the eccentricity of the hyperbola and is denoted by e,

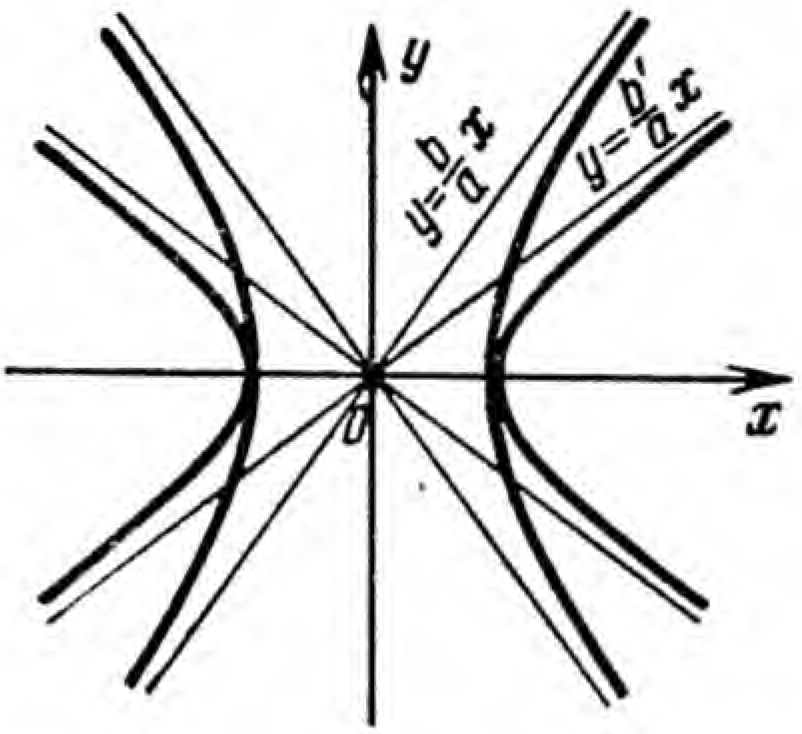

Fig. 43.

$$e = \frac{c}{a}.$$

Since $c > a$, $e > 1$.

From formula (XXXIV) it follows that $c = \sqrt{a^2 + b^2}$. Substituting this value of c into the eccentricity formula, we get

$$\boxed{e = \frac{\sqrt{a^2+b^2}}{a}} \qquad \text{(XXXVII)}$$

It follows from this formula that with a constant real axis $2a$ and decreasing imaginary axis $2b$, the eccentricity of the hyperbola approaches unity. At the same time, the smaller the value of b, the smaller the angle formed by the asymptotes $\left(\text{the ratio } \frac{b}{a} = \tan\varphi\right)$ and the more compressed the hyperbola in the vertical direction (Fig. 43).

2°. In the derivation of the equation of the hyperbola (Sec. 27) we obtained equality (2):

$$\frac{c}{a}x - a = \sqrt{(x-c)^2 + y^2}.$$

In it, $\sqrt{(x-c)^2 = y^2} = \overline{MF}$, which represents the radius vector of point M of the hyperbola. We denote it by r; the ratio $\frac{c}{a} = e$. Then equality (2) takes the form

$$\boxed{r = ex - a}$$

Denoting the second radius vector by r_1, $\overline{MF_1} = r_1$, we have

$$\boxed{r_1 = ex + a}$$

This formula shows that r and r_1 will be positive or negative as x is positive or negative. But since r and r_1 represent the *lengths*

of the radii vectors, their numerical values should be positive. As a result, when determining r and r_1 of points of the hyperbola with negative abscissas, it is necessary to take the absolute values of the right sides of the formulas for r and r_1.

Sec. 32. Equilateral Hyperbola

A hyperbola is called *equilateral* if its semiaxes are equal, $b=a$. Putting a for b in equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we get the equation of an equilateral hyperbola:

$$\boxed{x^2-y^2=a^2} \qquad \text{(XXXVIII)}$$

Putting $b=a$ in the equations of the asymptotes (XXXVI), we get the equations of the asymptotes of an equilateral hyperbola:

$$y=x \quad \text{or} \quad y=-x.$$

These asymptotes are the bisectors of the angles between the coordinate axes and, consequently, are perpendicular to each other, since if $\tan\varphi=\pm 1$, $\varphi=45^\circ$ or 135°. The eccentricity of any equilateral hyperbola is a constant,

$$e=\frac{\sqrt{a^2+a^2}}{a}=\sqrt{2}.$$

Sec. 33. Solved Examples on the Ellipse and Hyperbola

1°. Find the equation of the ellipse which has major axis $=10$ and eccentricity $e=0.8$.

Solution. To write the equation it is necessary to know the semiaxes a and b. It is given that $2a=10$ and $e=\frac{c}{a}=0.8$.

Hence, $a=5$, $c=0.8\cdot a=4$.

From the formula $b^2=a^2-c^2$ we find b^2:

$$b^2=5^2-4^2=9.$$

Putting the values $a^2=25$ and $b^2=9$ in equation (XXXI), we get the equation of the ellipse:

$$\frac{x^2}{25}+\frac{y^2}{9}=1 \quad \text{or} \quad 9x^2+25y^2=225.$$

2°. Find the equation of a hyperbola whose asymptotes are $y=\pm\frac{3}{5}x$ and which passes through the point $(-5, 2)$.

Solution. Comparing the given equations of the asymptotes with $y=\pm\frac{b}{a}x$, we conclude that $\frac{b}{a}=\frac{3}{5}$.

Since the hyperbola passes through point $(-5, 2)$, we can put these values in the equation of the hyperbola (XXXV) in place of the current coordinates and obtain

$$\frac{25}{a^2} - \frac{4}{b^2} = 1.$$

We get the following system of equations:

$$\frac{b}{a} = \frac{3}{5} \text{ and } \frac{25}{a^2} - \frac{4}{b^2} = 1.$$

We solve for a and b; from the first equation we find that $\frac{1}{a} = \frac{3}{5b}$ and substitute this into the second equation to get $\frac{9}{b^2} - \frac{4}{b^2} = 1$. Whence $b^2 = 5$. Substituting 5 for b^2 into the second equation, we get $\frac{25}{a^2} - \frac{4}{5} = 1$. Hence, $a^2 = \frac{125}{9}$. Putting these values of a^2 and b^2 into equation (XXXV), we get the required equation of the hyperbola:

$$\frac{x^2}{\frac{125}{9}} - \frac{y^2}{5} = 1 \quad \text{or} \quad 9x^2 - 25y^2 = 125.$$

3°. From the equation of the ellipse $5x^2 + 9y^2 = 180$ find its axes and eccentricity.

Solution. We reduce the given equation to the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

To do this we divide it by 180:

$$\frac{5x^2}{180} + \frac{9y^2}{180} = 1 \quad \text{or} \quad \frac{x^2}{36} + \frac{y^2}{20} = 1.$$

Whence

$$a = 6, \quad b = \sqrt{20} = 2\sqrt{5}.$$

Eccentricity:

$$e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{\sqrt{36 - 20}}{6} = \frac{4}{6} = \frac{2}{3}.$$

Answer. Axes: 12 and $4\sqrt{5}$, eccentricity: $\frac{2}{3}$.

Sec. 34. Parabola

Definition: *The parabola is the locus of points, in a plane, equidistant from a given point (the focus) and a given straight line (the directrix).*

By definition, if F is a given point, called the focus of the parabola (Fig. 44), KL is a given straight line called the **directrix**,

M is some point of the parabola, and MN is a perpendicular dropped from M on the straight line KL, then

$$\overline{MF} = \overline{MN}.$$

Sec. 35. Equation of a Parabola

1°. Let the line passing through the focus F perpendicular to the directrix KL be taken as the axis Ox (Fig. 44) and let the midpoint of AF (the distance of the focus F from the directrix KL) be the coordinate origin O. Let the length of AF be denoted by p,

$$\overline{AF} = p.$$

With these axes, the focus F has coordinates $\left(\frac{p}{2}, 0\right)$, point $M(x, y)$ and point $N\left(-\frac{p}{2}, y\right)$.

$$\overline{MF} = \sqrt{\left(x-\frac{p}{2}\right)^2 + y^2},$$

$$\overline{MN} = \sqrt{\left(x+\frac{p}{2}\right)^2}.$$

Since $\overline{MF} = \overline{MN}$,

$$\left(x-\frac{p}{2}\right)^2 + y^2 = \left(x+\frac{p}{2}\right)^2.$$

Whence

$$x^2 - px + \frac{p^2}{4} + y^2 = x^2 + px + \frac{p^2}{4}.$$

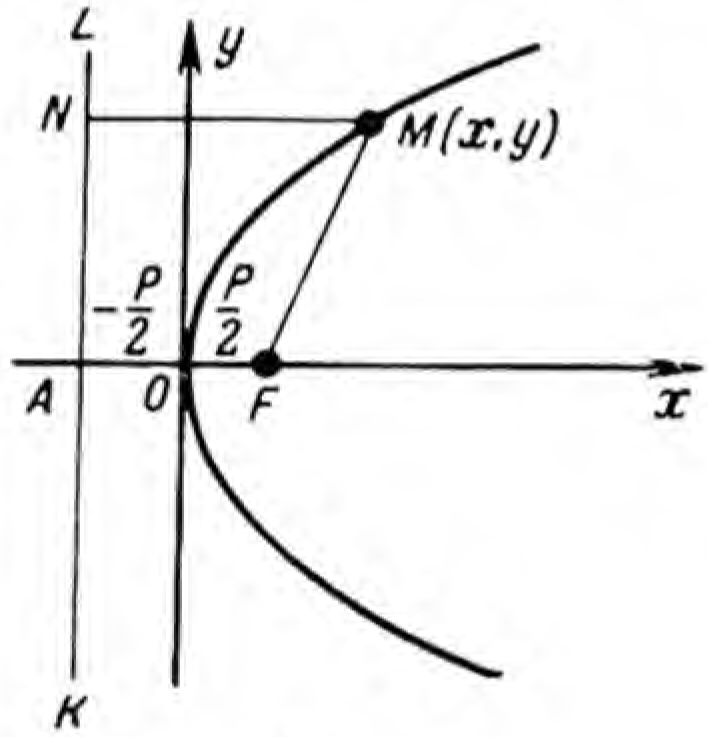

Fig. 44.

Eliminating x^2 and $\frac{p^2}{4}$ and transposing $-px$ to the right-hand side, we get the canonical equation of the parabola:

$$\boxed{y^2 = 2px} \qquad \text{(XXXIX)}$$

p is called the parameter of the parabola.

2°. The distance of the point M from the focus F is called the *radius vector* of M. Putting the radius vector of M equal to r, we get

$$r = x + \frac{p}{2},$$

since $r = \overline{MF}$, and $\overline{MF} = \overline{MN} = x + \frac{p}{2}$.

Sec. 36. Investigating the Forms of the Parabola from Its Equation

1°. From the equation of the parabola (XXXIX) we have

$$y = \pm \sqrt{2px}.$$

Since p is taken to be a positive number, the values of y can be real only for $x \geqslant 0$. If $x = 0$, $y = \pm \sqrt{2p \cdot 0} = 0$. The origin $O(0, 0)$ lies on the parabola and is called its *vertex*. $y^2 = 2px$ is an equation referred to the vertex of the parabola.

To each value of x in the interval $0 < x < +\infty$ there correspond two points of the parabola situated on either side of the axis Ox and at a distance from it equal to the absolute value of y, since to each value of x in this interval there correspond two values of y equal in magnitude and opposite in sign. The parabola is a curve symmetric about the axis Ox, which is called the *axis of the parabola*.

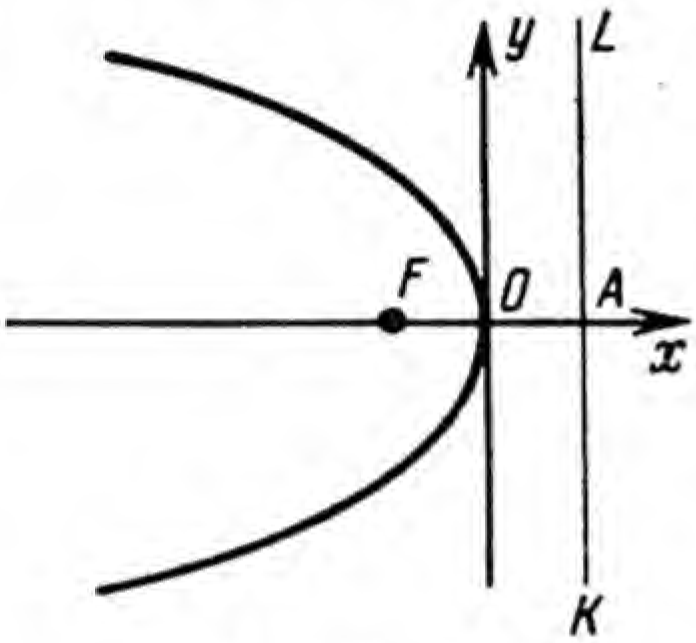

Fig. 45.

When x increases from 0 to $+\infty$, y increases in absolute value from 0 to ∞. The parabola is an infinite open curve such that we can always find a point on it with abscissa and ordinate as large as desired in absolute value, The parabola has the general form shown in Fig. 44.

2°. The distance from the vertex of the parabola O to its focus F is called the *focal length of the parabola*. The focal length of the parabola $OF = \frac{p}{2}$.

The directrix of the parabola is perpendicular to its axis; the equation of the directrix: $x = -\frac{p}{2}$.

3°. If $p < 0$ in equation $y^2 = 2px$, the ordinates of points of the parabola have real values only for negative abscissas. In this case the parabola is situated to the left of the axis Oy, and its directrix, to the right of the axis Oy (Fig. 45).

4°. If we interchange the moving coordinates x and y in the equation of the parabola, we obtain

$$\boxed{x^2 = 2py} \qquad \text{(XL)}$$

The axis of symmetry of this parabola is the axis Oy. When $p > 0$, the parabola lies above the x-axis (Fig. 46) and when $p < 0$, below the x-axis (Fig. 47).

It is worth remembering that the coordinate axis coincident with that moving coordinate which appears in the equation of the parabola only in the first degree serves as the axis of the parabola.

5°. It follows from $x^2=2py$ that

$$y=\frac{1}{2p}x^2.$$

Assuming $\frac{1}{2p}=a$, we get

$$y=ax^2,$$

which is the equation of the parabola we know from algebra.

6°. Note that the parabola has no centre.

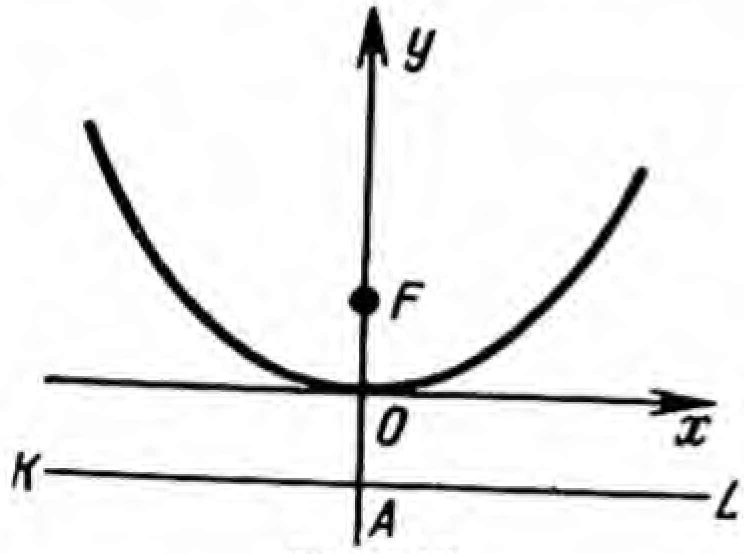

Fig. 46.

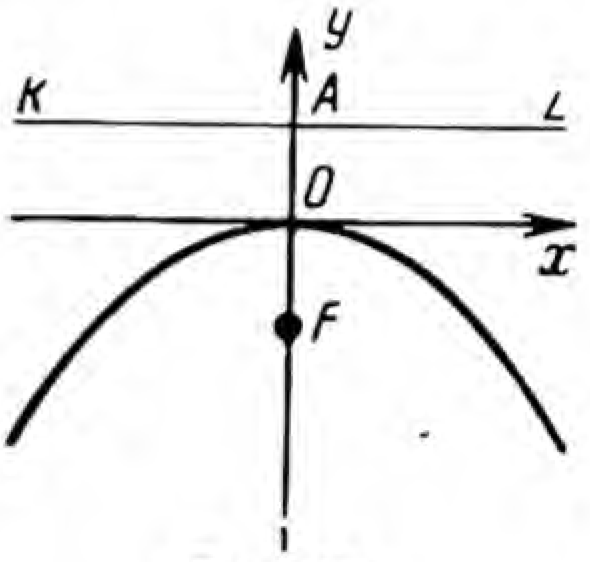

Fig. 47.

Sec. 37. Plotting a Parabola

1°. *By continuous motion* (Fig. 48).

We are given that p is the parameter of the parabola. Draw two mutually perpendicular straight lines KL and Ox and take one of them (KL) as the directrix and the other (Ox) as the axis

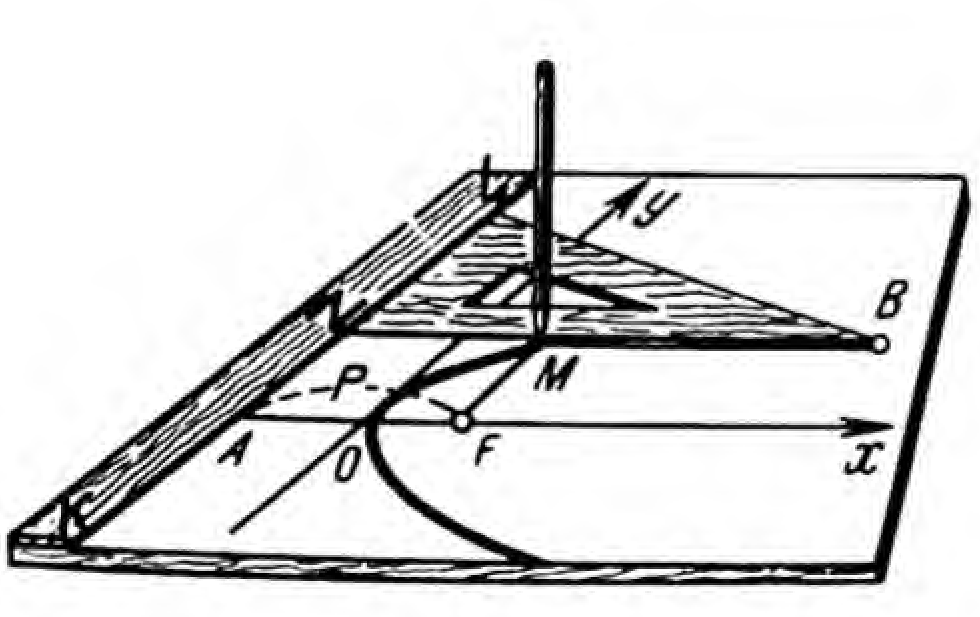

Fig. 48.

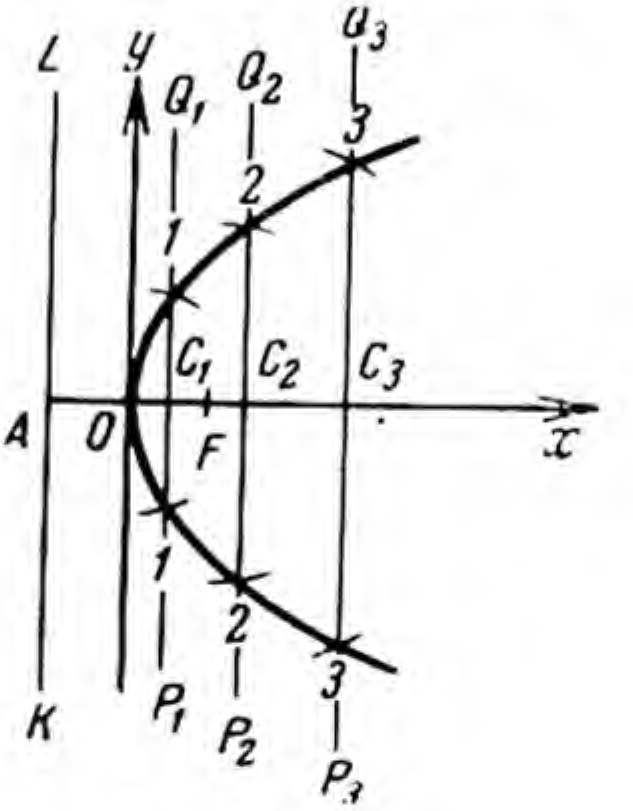

Fig. 49.

of the parabola. Using a compass, mark a distance AF, equal to p, along Ox from the directrix KL. This is the focus F. Lay a ruler along the directrix. Lay a set-square with its smaller side along

the ruler. Tie one end of a piece of thread to the vertex of the acute angle, B, opposite the smaller side of the set-square. Tie the other end of the thread to the focus F. [The length of the thread BF should be equal to the bigger side of the set-square.] Pulling the thread taut with the point of a pencil as shown in Fig. 48, draw a curve by moving the pencil so that the thread remains taut and the point of the pencil touches the side of the set-square and the set-square touches the ruler (Fig. 48). Every point M of this curve belongs to the parabola because $\overline{MN}=\overline{MF}$.

2°. *By points* (Fig. 49). Since the distance of any point of a parabola from the directrix is equal to the radius vector r and

$$r=x+\frac{p}{2},$$

a method can be devised to plot the points of the parabola. Construct, as shown in the preceding case, the directrix, axis, focus and vertex of the parabola. Draw a straight line P_1Q_1 parallel to the directrix so that it intersects the axis of the parabola at point C_1. Now, with focus F as the centre, strike arcs above and below the axis Ox so as to obtain their points of intersection with the line P_1Q_1. The radius of the arcs should be equal to AC_1 — the distance of the line P_1Q_1 from the directrix KL. These points of intersection (labelled *1* in Eig. 49) lie on the parabola, since for each of them

$$r=AC_1=\frac{p}{2}+x.$$

Drawing other straight lines P_2Q_2, P_3Q_3, etc., parallel to the directrix KL and performing the aforesaid operations with respect to each of them, we obtain any number of points (two in each case) all of which lie on the parabola (in the figure they are labelled by the numbers *2*, *3*, etc.).

Note that the points C_1, C_2, C_3, ... must be more closely spaced as they approach the vertex O and more widely spaced to the right of the vertex along the axis Ox.

When a sufficient number of points have been constructed, draw through them — by hand or with the help of a curved ruler — a smooth and continuous curve: the parabola.

Sec. 38. Formulas for the Transformation of Coordinates

1°. *Parallel translation of coordinate axes.*

If we have two systems of coordinates xOy and $x'O'y'$ (Fig. 50) in which the origin O' of the second system has the coordinates (a, b) in the first system and the directions of the axes in the systems are the same, i.e., the axis $O'x'$ is parallel to the axis Ox

and $O'y'$ is parallel to Oy with positive directions of the corresponding axes the same, then it is possible, knowing the coordinates of M in one system, to find the coordinates of this point in the other system.

Let M have coordinates (x', y') in the $x'O'y'$ system and (x, y) in the xOy system. Then from Fig. 50 (on the basis of rule 3°, Sec. 3) we have

$$\boxed{x' = x - a, \quad y' = y - b} \qquad \text{(XLI)}$$

2°. *Rotation of Coordinate Axes.*

Let M (Fig. 51) have coordinates $x = ON$ and $y = NM$ in the xOy system. Let the xOy system be rotated around the origin O

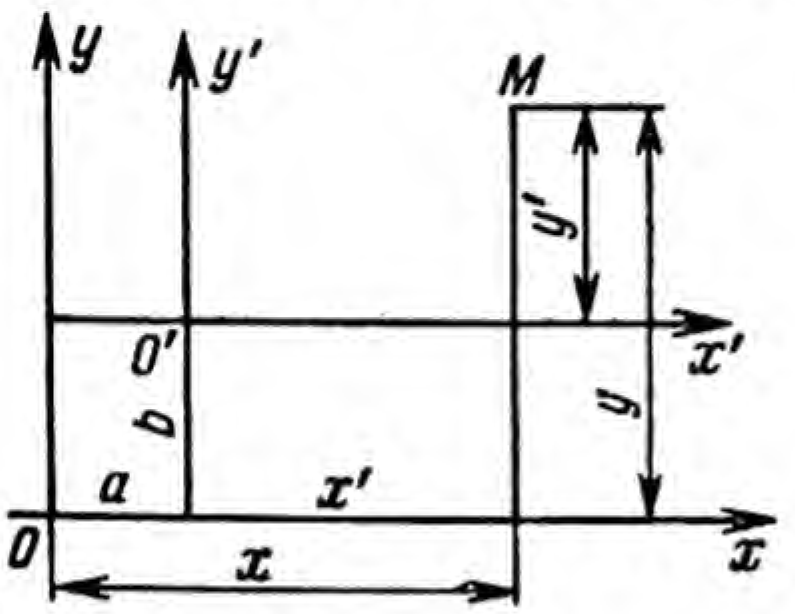

Fig. 50.

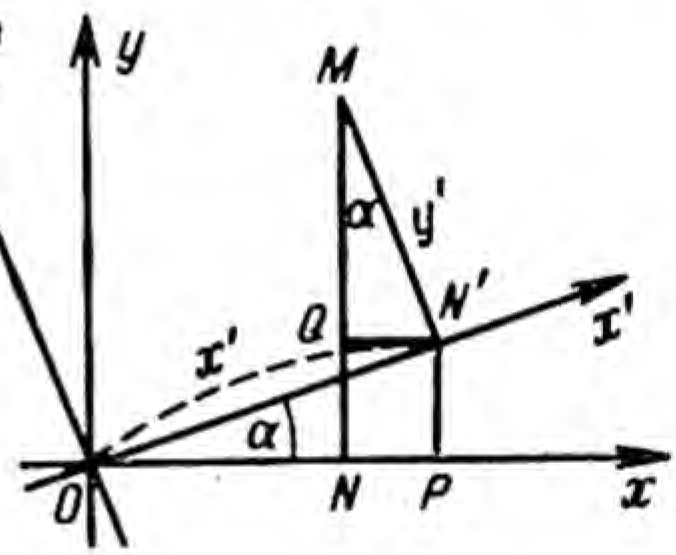

Fig. 51.

so that the new position of Ox' (the axis of abscissas) makes an angle α with its original position, Ox. This angle is called the angle of rotation; it is positive if the rotation of Ox is counterclockwise, and negative if the rotation is clockwise.

Let x' and y' be the coordinates of M along the new axes ($x' = ON'$, $y' = N'M$) and let us express the original coordinates x, y in terms of the new ones x', y' thus:

From Fig. 51

$x = ON = OP - NP =$	$y = NM = NQ + QM =$
$= OP - QN'$,	$= PN' + QM$,
from $\triangle\, OPN'$	
$OP = x' \cdot \cos\alpha$,	$PN' = x' \cdot \sin\alpha$;
from $\triangle\, QN'M$	
$QN' = y' \sin\alpha$.	$QM = y' \cdot \cos\alpha$.

Hence

$$\boxed{x = x'\cos\alpha - y'\sin\alpha; \quad y = x'\sin\alpha + y'\cos\alpha} \qquad \text{(XLII)}$$

Sec. 39. Equation of the Parabola in Parallel Translation of the Coordinate Axes

1°. Let the vertex of a parabola be at the point $O'(a, b)$, and let the axis of the parabola be parallel to the y-axis (Fig. 52). Imagine a system of coordinates $x'O'y'$ where the origin is the vertex of the parabola $O'(a, b)$, the axis $O'y' \parallel Oy$ and coincides with the axis of the parabola, and the axis $O'x' \parallel Ox$. In the $x'O'y'$ coordinate system, the equation of the parabola is

$$x'^2 = 2py';$$

in the xOy system it is

$$\boxed{(x-a)^2 = 2p(y-b)} \qquad \text{(XLIII)}$$

since for every point M of the parabola we have by formula (XLI)

$$x' = x - a, \quad y' = y - b.$$

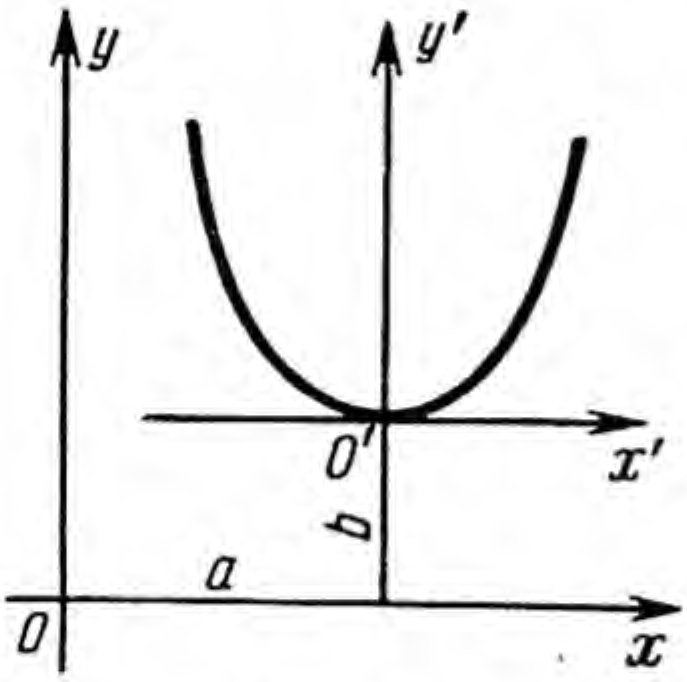

Fig. 52.

If the vertex of the parabola is $O'(a, b)$ and its axis $O'x'$ is parallel to the axis Ox, the equation of the parabola will be

$$\boxed{(y-b)^2 = 2p(x-a)} \qquad \text{(XLIV)}$$

2°. Solving equation XLIII for y, we get

$$y = \frac{1}{2p}(x-a)^2 + b$$

or

$$y = \frac{1}{2p}x^2 - \frac{a}{p}x + \frac{a^2}{2p} + b.$$

Assuming $\frac{1}{2p} = A$, $-\frac{a}{p} = B$, $\frac{a^2}{2p} + b = C$, we obtain

$$\boxed{y = Ax^2 + Bx + C} \qquad \text{(XLV)}$$

i.e., the parabola $(x-a)^2 = 2p(y-b)$ is the graph of the quadratic function $y = Ax^2 + Bx + C$.

3°. *The second-degree equation in the coordinates x, y:*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

describes a parabola with axis parallel to the y-axis (or parallel to the x-axis) if the second and third terms (or the second and first terms) are absent from the equation.

This is a necessary condition. Indeed, a parabola with axis parallel to the y-axis is expressed by equation (XLV):

$$y = Ax^2 + Bx + C = 0 \quad \text{or} \quad Ax^2 + Bx - y + C = 0,$$

i.e., by a general second-degree equation in x and y in which the second and third terms are absent.

This condition is sufficient. Indeed, dividing the equation

$$Ax^2 + Dx + Ey + F = 0$$

by A we get

$$x^2 + \frac{D}{A}x + \frac{E}{A}y + \frac{F}{A} = 0,$$

and expressing the quotient obtained as

$$x^2 + \frac{D}{A}x = -\frac{E}{A}y - \frac{F}{A},$$

we complete the square in $x^2 + \frac{D}{A}x$,

$$x^2 + 2x\frac{D}{2A} + \frac{D^2}{4A^2} = -\frac{E}{A}y - \frac{F}{A} + \frac{D^2}{4A^2}.$$

After rearranging the right side of the equation,

$$\left(x + \frac{D}{2A}\right)^2 = -\frac{E}{A}y + \frac{D^2 - 4AF}{4A^2},$$

$$\left(x + \frac{D}{2A}\right)^2 = -\frac{E}{A}\left(y - \frac{D^2 - 4AF}{4AE}\right),$$

we get an equation which represents a parabola with axis parallel to the y-axis, vertex with coordinates

$$a = -\frac{D}{2A}, \qquad b = \frac{D^2 - 4AF}{4AE},$$

and parameter

$$p = -\frac{E}{2A}.$$

Sec. 40. Equation of an Equilateral Hyperbola Referred to the Asymptotes

The equation of an equilateral hyperbola with respect to the system xOy (Fig. 53) is

$$x^2 - y^2 = a^2.$$

Rotate the xOy coordinate system about the origin O through the angle $\alpha = -45°$. Then the asymptotes $y = -x$ and $y = +x$ will be the new coordinate axes Ox' and Oy'. By formulas (XLII)

we have for each point of the equilateral hyperbola:

$$x = x' \cdot \cos(-45°) - y' \cdot \sin(-45°) = x' \cdot \frac{1}{\sqrt{2}} + y' \cdot \frac{1}{\sqrt{2}} = \frac{x'+y'}{\sqrt{2}},$$

$$y = x' \cdot \sin(-45°) + y' \cdot \cos(-45°) = -x' \cdot \frac{1}{\sqrt{2}} + y' \cdot \frac{1}{\sqrt{2}} = \frac{y'-x'}{\sqrt{2}},$$

where x', y' are the new coordinates of point (x, y) of the hyperbola.

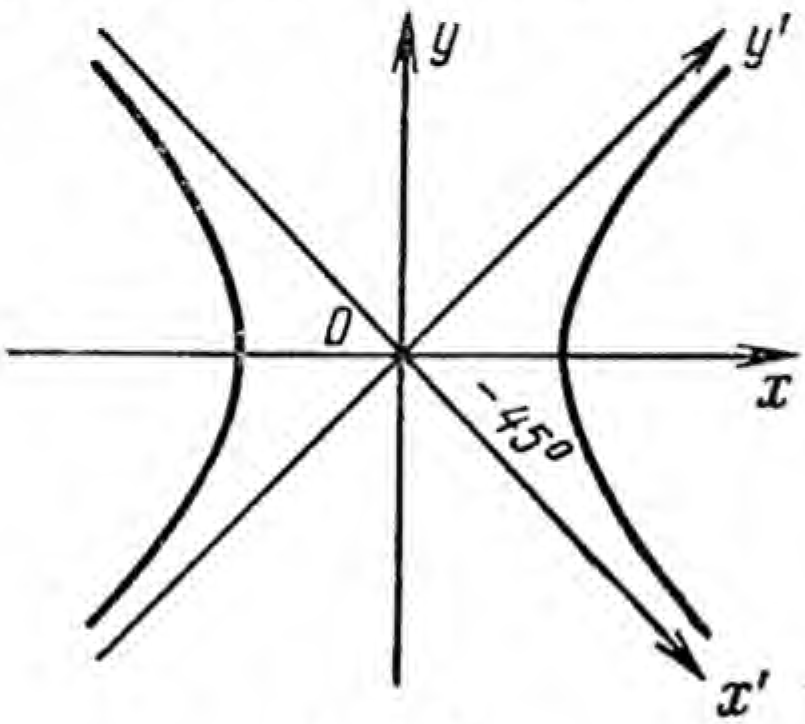

Fig. 53.

Substituting these values of x, y into equation $x^2 - y^2 = a^2$, we have

$$\frac{(x'+y')^2}{2} - \frac{(y'-x')^2}{2} = a^2$$

whence we get $4x'y' = 2a^2$,

$$\boxed{x'y' = \frac{a^2}{2}} \qquad \text{(XLVI)}$$

This is the equation of an equilateral hyperbola when the asymptotes of the hyperbola serve as the coordinate axes.

Denoting $\frac{a^2}{2}$ by k and dropping the primes on the current coordinates x', y', we get the form

$$xy = k.$$

Thus, an equilateral hyperbola with its equation referred to the asymptotes is a graph of inverse proportionality.

Sec. 41. Solved Examples

1°. Find the equation of a parabola with vertex in the origin if the coordinates of the focus are $(-2, 0)$.

Solution. It is given that the focus lies on the x-axis and the vertex at the origin. Therefore, the equation of the parabola is of the form:

$$y^2 = 2px.$$

The focal length of the parabola $OF = \frac{p}{2} = -2$. Hence, $2p = -8$ and the required equation is

$$y^2 = -8x.$$

2°. Find the equation of a parabola, knowing the equation of the directrix, $y = -3$, and the coordinates of the focus $(4, 1)$.

Solution. It is given that the directrix is parallel to the x-axis. Therefore, the axis of the parabola is parallel to the y-axis and its

equation is $x=4$. The point of intersection of the directrix and the axis of the parabola is $A(4, -3)$. The parameter p is the distance from $A(4, -3)$ to $F(4, 1)$, which is found by formula (I) in Sec. 3:

$$p=1-(-3)=4.$$

The vertex $O(a, b)$ is the mid-point of the segment AF. By formula V, Sec. 4:

$$a=\frac{4+4}{2}=4, \quad b=\frac{-3+1}{2}=-1.$$

The equation of the parabola has the form

$$(x-a)^2=2p(y-b).$$

Putting the values of a, b, p in the above equation, we get

$$(x-4)^2=8(y+1) \text{ or } x^2-8x-8y+8=0.$$

3°. Determine the coordinates of the vertex and the magnitude of the parameter of the parabola whose equation is

$$y^2+5x-6y+14=0.$$

Also find the coordinates of its focus and the equation of the directrix.

Solution. Shift the coordinate origin to the vertex of the parabola $O'(a, b)$ keeping the direction of the axes unchanged. Let us denote the new system of coordinates by $x'O'y'$. It follows from formula XLI that

$$\boxed{x=x'+a, \quad y=y'+b} \tag{XLIa}$$

Introducing these expressions of x and y into the given equation, we get an equation of the parabola referred to the vertex:

$$(y'+b)^2+5(x'+a)-6(y'+b)+14=0,$$

or

$$y'^2+2by'+b^2+5x'+5a-6y'-6b+14=0,$$

or

$$y'^2+5x'+(2b-6)y'+(5a-6b+b^2+14)=0. \tag{1}$$

But the equation of the parabola referred to the vertex O' has the form

$$y'^2=2px' \text{ or } y'^2-2px'=0. \tag{2}$$

For equations (1) and (2) to be identically equal it is necessary that

1) $2b-6=0$, 2) $5a-6b+b^2+14=0$ and 3) $5=-2p$.

We find from the first condition that $b=3$. Substituting this value of b into the second condition, we get

$$5a-6\cdot 3+3^2+14=0, \quad a=-1.$$

It follows from the third condition that $p=-\frac{5}{2}$.

Hence, the given parabola has the vertex $O'(-1, 3)$ and the parameter $p=-\frac{5}{2}$. The negative sign of the parameter indicates that though the axis of the parabola is parallel to the x-axis it has a direction opposite to the positive direction of the x-axis.

In the $x'O'y'$ system with origin at the vertex of the parabola, the focus of the parabola has coordinates $\left(\frac{p}{2}, 0\right)$; and the vertex O' of the parabola has coordinates (a, b) in the xOy system. The coordinates of the focus in the given system xOy are obtained by formula (XLIa):

$$x_{foc}=\frac{p}{2}+a=-\frac{5}{4}-1=-2\frac{1}{4}, \quad y_{foc}=0+b=3.$$

This means that the given parabola has focus $F\left(-2\frac{1}{4}, 3\right)$.

The equation of the directrix in the system of axes with vertex at the origin $O'(a, b)$ is $x'=-\frac{p}{2}$. Using formula (XLIa) we obtain the equation of the directrix:

$$x=-\frac{p}{2}+a=-\left(-\frac{5}{4}\right)-1=\frac{1}{4}.$$

4°. The foregoing method of transforming a given equation into the canonical form may be used to obtain the parameters of the equation of any quadric curve, if only the equation does not have a term with xy.

Sec. 42. Quadric Curves as Conic Sections

1°. If an infinite straight line SA (Fig. 54) is translated in space so that all the time it passes through point S and slides along some curve $ACBD$, the surface thus generated by the straight line SA is called a *conic surface* or simply a *cone*. The curve $ACBD$ is, in this instance, called the *guiding line*; the straight line SA is called the *generating line* of the cone and the point S, the *vertex* of the cone. If the guiding line has a centre O, then the straight line SO, joining the vertex with the centre, is called the *axis* of the cone. A conic surface consists of two cavities extending on opposite sides of the vertex S to infinity.

We shall consider a *right circular cone*. It is obtained with a circle as the guiding curve $ACBD$, and the vertex S as a point on the perpendicular to the plane of the circle $ACBD$ through its centre; the perpendicular OS will thus be the axis of the cone.

The section of the cone cut by a plane passing through the axis SO is called the *axial section of the cone*, and angle ASB at the vertex S in the axial section is called the *angle of the axial section*.

2°. On a conic surface, take an arbitrary point Q (Fig. 54) not coinciding with the vertex S. Through this point make an axial section $ASQB$ and planes PQ, P_1Q, P_2Q, P_3Q perpendicular to the plane of the axial section $ASQB$. The following cases may occur:

1) the cutting plane PQ is perpendicular to the axis SO

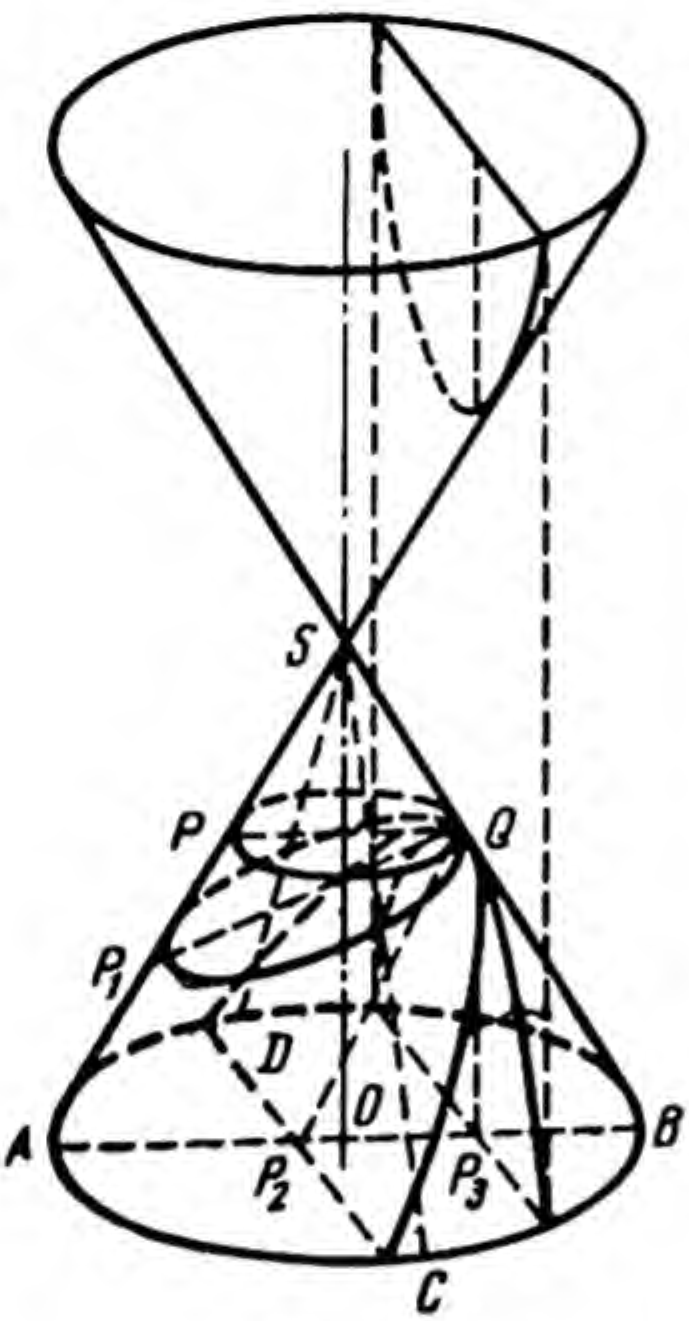

Fig. 54.

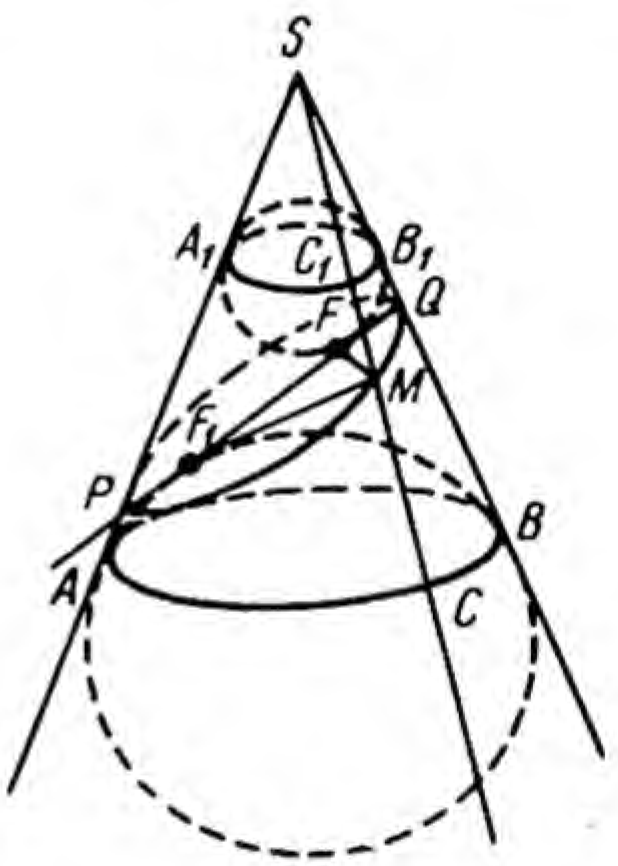

Fig. 55.

of the cone. The line of intersection of the plane with the conic surface is a circle;

2) the plane P_1Q intersects each of the generating lines in one cavity of the cone at a finite distance from its vertex S (this will take place so long as $\angle P_1QB$ is larger than $\angle ASB$); the line of intersection in this case is a closed curve, an ellipse;

3) the plane P_2Q is parallel to one of the generating lines, e. g., SA, ($\angle P_2QB = \angle ASB$); it intersects other generating lines on one side of the vertex S; in this case the line along which the cutting plane intersects the conic surface is an open curve with one point at infinity (parabola);

4) the plane P_3Q is parallel to two generating lines of the cone (SC and SD in Fig. 54); in this case the plane intersects both cavities of the cone ($\angle P_3QB < \angle ASB$) and in each cavity the line of intersection forms an open curve, a branch of a hyperbola.

Thus when a plane cuts a conic surface we obtain a quadric curve. For this reason, quadric curves are called *conic sections*.

Let us prove that the conic section will be an ellipse if the cutting plane PQ (Fig. 55) intersects every generating line of the cone on one side of its vertex S.

Proof. Let us inscribe in the cone a sphere on either side of the cutting plane PQ so that the plane PQ is tangential to both spheres. Let the points of contact be F and F_1.

Since the tangents to a sphere from a given point are always equal,

a) all the points of contact, with the cone, of each of the inscribed spheres lie in a single plane perpendicular to the axis of the cone, viz., in the plane $A_1B_1C_1$ for the upper sphere and in the plane ABC for the lower sphere;

b) the segments of the generating lines of the cone contained between these planes are equal, i.e.,

$$\overline{AA_1} = \overline{BB_1} = \overline{CC_1} = \ldots = \text{const.} \tag{1}$$

Let us take an arbitrary point M on the line of the cone cut by the plane PQ and join M to points F and F_1. The segments MF and MF_1 are tangents to the respective spheres.

Through M draw the generating line SM which is tangent to one sphere at C and to the other sphere at C_1.

Thus, from M two tangents are drawn to the lower sphere and to the upper sphere. The tangents to the lower sphere are MF_1 and MC, where

$$\overline{MF_1} = \overline{MC}, \tag{2}$$

and those to the upper sphere are MF and MC_1, where

$$\overline{MF} = \overline{MC_1}. \tag{3}$$

Adding equation (2) to (3), we get

$$\overline{MF} + \overline{MF_1} = \overline{MC} + \overline{MC_1} = \overline{CC_1}.$$

By equation (1) segment CC_1 does not change in length on translation of M along the line of the cone cut by the plane PQ. Hence this line has the property that the sum of the distances of any point M from the points F and F_1 is constant. Consequently, this line is an ellipse with foci at points F and F_1.

Similarly it may be shown that the parabola and the hyperbola are also conic sections.

B. ELEMENTS OF DIFFERENTIAL CALCULUS

CHAPTER V

THEORY OF LIMITS

Sec. 43. Absolute Value and Its Properties

Let us agree from now on to use the term "number" in the sense of "real number".

1°. **Definition.** *The absolute value of a positive number is the positive number itself; the absolute value of a negative number is the positive value of that number; the absolute value of zero is taken to be equal to zero.*

Absolute value is signified by placing the number within two vertical bars.

Thus, $|7|=7$, $|-3|=3$, $|0|=0$. If the number a is positive, $|a|=a$; if a is negative, $|a|=-a$.

2°. **Properties.** 1. *The absolute value of a sum of numbers is no greater than the sum of the absolute values of the terms of the sum,* i. e.,

$$|a+b| \leqslant |a|+|b|.$$

Indeed, if numbers a and b are both positive or both negative, their sum is obtained by adding their absolute values and putting the result under the common sign of the terms of the sum. Thus the absolute value of a sum is equal to the sum of the absolute values of the terms. For example, $a=-3$, $b=-7$:

$$|(-3)+(-7)|=|-10|=10$$

and

$$|-3|+|-7|=3+7=10,$$

$$|(-3)+(-7)|=|-3|+|-7|.$$

If a and b are opposite in sign, their sum is obtained by subtracting the smaller absolute value from the greater and putting the sign of the term with the greater absolute value. Therefore, the absolute value of the sum is less than the sum of the absolute

values of the terms. For example, the absolute value of the sum of numbers $|(-7)+(+3)|=|-4|=4$, while the sum of the absolute values of these numbers is $|-7|+|+3|=7+3=10$; i.e.,

$$|(-7)+(+3)|<|-7|+|+3|.$$

The property which has been proved can be extended to any constant number of terms, i.e.,

$$|a+b+\ldots+k|\leqslant|a|+|b|+\ldots+|k|.$$

2. *The absolute value of the difference of two numbers is greater than, or equal to, the difference between the absolute values of these numbers*, i.e.,

$$|a-b|\geqslant|a|-|b| \quad \text{or} \quad |a-b|\geqslant|b|-|a|.$$

Indeed, let

$$a-b=c.$$

Hence,

$$a=b+c.$$

From the preceding arguments,

$$|a|\leqslant|b|+|c|.$$

Solving this inequality for $|c|$, we get

$$|c|\geqslant|a|-|b|$$

or

$$|a-b|\geqslant|a|-|b|.$$

The absolute value of a number does not change with change of sign. Hence it is always true that

$$|a-b|=|b-a|.$$

But from the foregoing

$$|b-a|\geqslant|b|-|a|.$$

Hence the following inequality is also true:

$$|a-b|\geqslant|b|-|a|.$$

Note that the difference $|a|-|b|$ or $|b|-|a|$ can be negative.

3. *The absolute value of a product is equal to the product of the absolute values of the factors*, i.e.,

$$|a\cdot b|=|a|\cdot|b|.$$

4. *The absolute value of a quotient is equal to the quotient obtained by the division of the absolute value of the dividend by the absolute value of the divisor*, i.e.,

$$\left|\frac{a}{b}\right|=\frac{|a|}{|b|}.$$

5. *The absolute value of a power with positive integral exponent is equal to the same power of the absolute value of the base*, i.e.,

$$|a^n| = |a|^n.$$

Properties 3 to 5 follow directly from the properties of multiplication and division.

Sec. 44. Infinitely Small Quantity (Infinitesimal)

In engineering and in natural sciences, one particular kind of variable plays an especially important role, namely, the infinitely small variable (infinitesimal).

1°. **Definition.** *A variable is said to be infinitely small, if, from a certain moment onwards in its process of variation, the absolute magnitudes of all subsequent values of the variable ultimately become and remain smaller than any positive number* ε.

2°. **Examples.** 1. The side of a regular inscribed polygon is an infinitely small quantity when the number of sides of the polygon is doubled indefinitely, since in this process the side can be made as small as desired.

2. The fraction $\frac{1}{x}$ is an infinitely small quantity when the absolute value of x increases indefinitely $\left(\text{e. g.}, \frac{1}{x} = \frac{1}{10}, \frac{1}{100}, \frac{1}{1,000}, \ldots \text{ or } \frac{1}{x} = \frac{1}{-10}, \frac{1}{-100}, \frac{1}{-1,000}, \ldots\right)$. Indeed, no matter how small the given positive number ε, a time will come, as $|x|$ increases indefinitely, when, from this instant onwards, $|x|$ will be greater than $\frac{1}{\varepsilon}$:

$$|x| > \frac{1}{\varepsilon} \text{ and } \left|\frac{1}{x}\right| < \varepsilon.$$

3. The fraction $\frac{p}{x}$, where p is constant and x increases indefinitely, is an infinitely small quantity, since it is only necessary to take $|x| > \frac{|p|}{\varepsilon}$ to obtain $\left|\frac{p}{x}\right| < \varepsilon$.

3°. We shall denote infinitely small quantities, or infinitesimals, by the Greek letters α, β, γ, ...

By definition, α is an infinitesimal if

$$|\alpha| < \varepsilon$$

where ε is any given small positive number.

4°. An infinitesimal should not be confused with a small number. Any small number $c \neq 0$ is non-variable, and some other

positive number ε can always be found such that $|c|$ will not be less than ε.

Hence, *no small number c, not equal to zero, can be called an infinitesimal.*

Sec. 45. Variable Quantities, Bounded and Unbounded

1°. **Definition.** *The variable x is called a bounded quantity if, from and after a certain moment,*

$$|x| \leqslant m \tag{1}$$

where m is some positive number. Otherwise, it is called an unbounded quantity.

For an unbounded quantity it is impossible to choose a number m, for which (at a certain time) inequality (1) would be fulfilled; on the contrary, an unbounded variable can have values for which the inequality $|x| > m$ holds for any m.

2°. Any given number may be regarded as a bounded quantity.

3°. An infinitesimal α is necessarily a bounded quantity since, from and after a certain moment, its absolute value not only becomes less than some definite positive number m but less than *any* assigned small positive number ε,

$$|\alpha| < \varepsilon.$$

Sec. 46. Basic Properties of Infinitesimals

1°. *An infinitesimal* (infinitely small quantity) *remains an infinitesimal on a change of its sign.*

2°. **Theorem.** *If α and β are infinitesimals, their sum or difference is also an infinitesimal.*

Proof. Assume that to every moment of approach of one of the infinitesimals to zero there corresponds a definite value of each of the infinitesimals being considered.

(For example, when $\alpha = 0.1,\ 0.01,\ 0.001, \ldots$ β is, at the same moments, equal respectively to $-0.01,\ -0.0001,\ -0.000001, \ldots$).

In any case, no matter how differently the values of α and β may change in their approach to zero, there will be a time starting with which the absolute value of each of the infinitesimals will remain less than $\frac{\varepsilon}{2}$:

$$|\alpha| < \frac{\varepsilon}{2} \text{ and } |\beta| < \frac{\varepsilon}{2},$$

and, consequently, their sum will remain less than ε,

$$|\alpha| + |\beta| < \varepsilon.$$

But $|\alpha+\beta| \leqslant |\alpha|+|\beta|$ (Sec. 43, 2°). Hence

$$|\alpha+\beta| < \varepsilon.$$

Thus the sum $\alpha+\beta$ is an infinitesimal (Sec. 44, definition).

3°. Since the infinitesimal β remains an infinitesimal on change of sign, the difference $\alpha-\beta$, which is equal to the sum of the infinitesimals α and $-\beta$,

$$\alpha-\beta=\alpha+(-\beta)$$

is itself an infinitesimal.

Corollary. *The algebraic sum of several infinitesimals is itself an infinitesimal.*

For example, $\alpha+\beta-\gamma+\delta+\xi$ must be an infinitesimal since each operation of addition or subtraction of two infinitesimals results in another infinitesimal.

4°. **Theorem.** *The product of a bounded quantity x and an infinitesimal α is itself an infinitesimal.*

Proof. Assume that for every moment of time during the approach of α to zero there exist definite values of α as well as x. (For example, $x=1.95$, 1.995, $1.9995, \ldots$, when $\alpha=0.1$, 0.01, $0.001, \ldots$).

During the variation of α and x there will inevitably be a time, starting with which the following inequalities will always hold true:

$$|x| < m, \tag{1}$$

$$|\alpha| < \frac{\varepsilon}{m}, \tag{2}$$

where m is some definite positive number.

Multiplying (1) by (2) we find that from that time the product $|x|\cdot|\alpha| < \varepsilon$.

But $|x|\cdot|\alpha|=|x\cdot\alpha|$ (Sec. 43, Point 3).

Hence

$$|x\cdot\alpha| < \varepsilon.$$

Thus the product $x\cdot\alpha$ is an infinitesimally small quantity (by definition, Sec. 44).

Corollary 1. *The product of an infinitesimal and a constant is an infinitesimal*, since any constant may be regarded as a bounded quantity.

Corollary 2. *The product of several infinitesimals is an infinitesimal.*

An infinitesimal β is a bounded quantity (Sec. 45, 3°). Therefore the product $\alpha\cdot\beta$ is an infinitesimal.

Similarly the product $\alpha\cdot\beta\cdot\gamma=(\alpha\cdot\beta)\cdot\gamma$, where γ is a third infinitesimal, is also an infinitesimal, and so on.

Corollary 3. *A positive integral power of an infinitesimal is an infinitesimal*, since

$$\alpha^n = \overbrace{\alpha \cdot \alpha \ldots \alpha}^{n \text{ times}}.$$

5°. **Note.** The quotient of two infinitesimals may not be an infinitesimal. For example, if

$$\alpha = 2\beta, \quad \text{then } \frac{\alpha}{\beta} = \frac{2\beta}{\beta} = 2.$$

It will be shown below (Sec. 47) that the quotient of two infinitesimals may be a bounded non-infinitesimal, an infinitesimal, or an infinitely large quantity.

Sec. 47. Infinitely Large Quantity

The terms "infinitely small quantity" and "infinitely large quantity" do not characterise the actual magnitude of a (variable) quantity but define the kind of variation of its numerical value.

1°. **Definition.** *A variable x is said to be infinitely large if, beginning from a certain time in the process of its variation, its absolute*

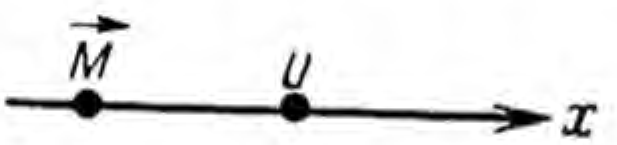

Fig. 56.

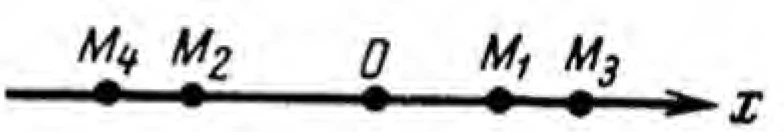

Fig. 57.

value becomes and remains greater than any positive number N, i.e., if beginning with a certain value of x, the following inequality holds:

$$|x| > N,$$

however large the positive number N.

Example. When the arc α increases from 0 to $\frac{\pi}{2}$, $\tan\alpha$ is positive and increases without bound; when the arc α decreases from 0 to $-\frac{\pi}{2}$, $\tan\alpha$ is negative and decreases without bound. In both cases, beginning with a certain value of the arc α, the absolute value of $\tan\alpha$ becomes and remains greater than any given positive number N, however large, i.e.,

$$|\tan\alpha| > N.$$

For this reason, $\tan\alpha$ is an infinitely large quantity both when α increases from 0 to $\frac{\pi}{2}$ and when it decreases from 0 to $-\frac{\pi}{2}$.

2°. The concept of an infinitely large quantity can be graphically illustrated by taking a point M (Fig. 56) with abscissa x moving along the axis Ox in some particular direction (to the left or to the right) or receding from the origin O to a larger and larger distance alternately to the left and right (Fig. 57)*.

Sec. 48. Relationship Between Infinitely Small and Infinitely Large Quantities

1°. *If x is an infinitely large quantity, then its inverse $\frac{1}{x}$ is an infinitely small quantity.*

Indeed, no matter how small a positive number ε, by the definition of an infinitely large quantity there will come a time when $|x|$ will exceed $\frac{1}{\varepsilon}=N$, i.e.,

$$|x|>N=\frac{1}{\varepsilon}.$$

But then

$$\left|\frac{1}{x}\right|<\varepsilon.$$

Consequently, $\frac{1}{x}$ is an infinitesimal (by definition, Sec. 44).

2°. *If α is an infinitesimal that never becomes zero, then its inverse $\frac{1}{\alpha}$ is an infinitely large quantity.*

Indeed, no matter how great a positive number N, there will come a time for the infinitesimal α, when $|\alpha|$ will be less than $\frac{1}{N}=\varepsilon$, i.e.,

$$|\alpha|<\varepsilon=\frac{1}{N}.$$

But then

$$\left|\frac{1}{\alpha}\right|>N.$$

Consequently, $\frac{1}{\alpha}$ is an infinitely large quantity (by definition, Sec. 47).

3°. **Note.** When studying the relationship between an infinitely small and an infinitely large quantity (2°) we take *the infinitely small quantity that never becomes zero*. If the infinitesimal α were permitted to have the value zero, the fraction $\frac{1}{\alpha}$ would cease to be a number since division by zero is impossible.

* For example, $x=(-1)^n \cdot n$, where n is a positive integer.

Let us examine *why it is impossible to divide by zero.* To find the quotient of a number a divided by b means to find a third number c, the product of which by the divisor b is equal to the dividend a, i.e.,

$$\text{if } \frac{a}{b}=c, \text{ then } c\cdot b=a.$$

But if the divisor $b=0$ and the dividend $a \neq 0$ and if we assume that

$$\frac{a}{0}=c,$$

then the product $0\cdot c$, whatever the value of c, is equal to zero, and not to the number a:

$$0\cdot c \neq a.$$

Therefore, *the quotient of* $\frac{a}{0}$ *cannot be taken to be any definite number.*

Neither is it possible to divide 0 *by* 0. Indeed, if we assume that the quotient of $\frac{0}{0}$ is the number c,

$$\frac{0}{0}=c,$$

then

$$c\cdot 0=0.$$

However, here c is not some one number, but any number, since the multiplication of any number by zero yields zero. For this reason, division of zero by zero is not considered.

Sec. 49. The Limit of a Variable Quantity

The student has encountered the concept of limit in geometry and algebra. Here we shall consider it in more detail.

1°. **Definition.** *A number a is called the limit of a variable quantity x if, in the course of variation, x approaches a so that beginning with a certain time the absolute value of the difference $x-a$ becomes and remains small without bound, i.e., less than any given positive number.*

We shall denote by the letter ε any given small positive number. By definition, for the number a to be the limit of x, the following inequality must be fulfilled beginning with a certain value of x:

$$|x-a|<\varepsilon.$$

2°. **Example 1.** Let the variable x take on consecutively the following values: $x_1=1.95$; $x_2=1.995$; $x_3=1.9995$, ..., $x_n=\overbrace{1.99\ldots 95}^{n\text{ times}}$; ...

The value of x approaches the number 2. Let us verify that this approach to 2 is without bound, i.e., that the absolute value of $x-a$ can be made less than any given small positive number.

We determine the absolute values of the difference between the values of x and the number 2:

$$|x_1-2|=|1.95-2|=|-0.05|=0.05;$$

$$|x_2-2|=|1.995-2|=|-0.005|=0.005 \text{ and so on}$$

$$|x_n-2|=|\overbrace{1.99\ldots95}^{n}-2|=|-\overbrace{0.00\ldots05}^{n}|=$$

$$=\overbrace{0.00\ldots05}^{n} \text{ and so on.}$$

Let us take, arbitrarily, some small positive number, say, 0.000001. The absolute value of the difference between the value x_n of the variable x and 2, equal to $\overbrace{0.00\ldots05}^{n}$, will at least be less than 0.000001, when $n \geqslant 6$.

We may take any positive number ε, not necessarily 0.000001. For 2 to be the limit of x it is necessary that, beginning with a certain value of the number n, the following inequality should invariably be fulfilled:

$$|x_n-2|<\varepsilon$$

or

$$\overbrace{0.00\ldots05}^{n}\ \frac{1}{2\cdot 10^n}<\varepsilon.$$

Hence, we should have

$$2\cdot 10^n>\frac{1}{\varepsilon},$$

$$10^n>\frac{1}{2\varepsilon},$$

$$n>\log\frac{1}{2\varepsilon},$$

which means that once n becomes greater than $\log\frac{1}{2\varepsilon}$ the inequality

$$|x-2|<\varepsilon$$

is invariably satisfied for all values of x.

Hence the limit of $x=2$.

3°. Example 2. *As the arc α tends to zero, the limit of $\sin\alpha$ is equal to zero.*

Proof. The variable $x=\sin\alpha$. For the value of the arc we take the number α expressed in radians.

Let the radius of the circle represent a unit of length. Then the length of the line of the sine, $\overline{BC}$ or $\overline{B_1C}$ in Fig. 58, is equal to $|\sin\alpha|$ and the length of the arc AB (or $\smile \overline{AB_1}$) is equal to $|\alpha|$.

Let us examine how α approaches zero either from some positive or some negative value, e. g., from some value of α the absolute magnitude of which is less than $\frac{\pi}{2}$,

$$|\alpha| < \frac{\pi}{2}.$$

Then

$$|\sin\alpha| \leqslant |\alpha|,$$

since the length of the half-chord BB_1 is always less than its corresponding half-arc BAB_1, and when the chord and the arc both convert to the point A, $\sin\alpha = \alpha = 0$.

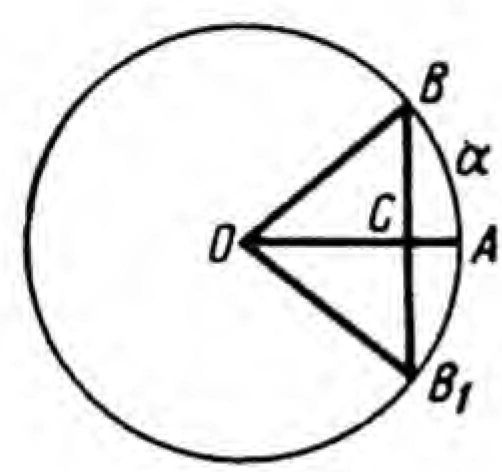

Fig. 58.

But if

$$|\sin\alpha| \leqslant |\alpha|,$$

then it must also be that

$$|\sin\alpha - 0| \leqslant |\alpha|.$$

Bringing the magnitude of arc α to zero, we obtain a value of α such that, beginning with it,

$$|\alpha| < \varepsilon,$$

however small the given positive number ε.

Hence it follows that

$$|\sin\alpha| < \varepsilon \text{ and } |\sin\alpha - 0| < \varepsilon$$

or the limit $\sin\alpha = 0$ as α tends to zero.

4°. **Example 3.** Let the variable $x = \frac{\sin\alpha}{\alpha}$. We shall prove that the limit of x is equal to zero when α is positive and increases indefinitely.

Proof. At no value of α can $\sin\alpha$ be greater than unity. Hence, replacing the numerator in $\frac{\sin\alpha}{\alpha}$ by unity, we get

$$\frac{|\sin\alpha|}{\alpha} \leqslant \frac{1}{\alpha}$$

(α is positive).

No matter how small the given positive number ε, there will come a time (as α increases without bound) such that, beginning with this time, α will be greater than $\frac{1}{\varepsilon}$:

$$\alpha > \frac{1}{\varepsilon} \text{ and } \frac{1}{\alpha} < \varepsilon.$$

Therefore,

$$\left|\frac{\sin\alpha}{\alpha}\right| < \varepsilon \text{ and } \left|\frac{\sin\alpha}{\alpha} - 0\right| < \varepsilon.$$

Consequently, the limit of x is zero.

5°. The expression "$\lim x$" means "limiting value of x".

By the definition of a limit, if, from and after a certain value of x, the following inequality is satisfied

$$|x-a|<\varepsilon,$$

then

$$\lim x = a.$$

If $|x-a|<\varepsilon$, it is said that the "variable x approaches a indefinitely" or "x tends to a", and the fact is symbolised by $x \longrightarrow a$.

Remember that the notation $x \longrightarrow a$ means that the limit of x is a.

The above studied limits of variables can thus be written as

$$1.\quad \lim_{n\to\infty} \overbrace{1.99\ldots95}^{n} = 2;$$

$$2.\quad \lim_{\alpha\to\infty} \sin\alpha = 0;$$

$$3.\quad \lim_{\alpha\to+\infty} \frac{\sin\alpha}{\alpha} = 0.$$

6°. If the variable α is an infinitesimal, its absolute value becomes less than any given positive number ε, however small,

$$|\alpha|<\varepsilon.$$

But then the absolute value of the difference between α and 0 is also less than ε,

$$|\alpha-0|<\varepsilon.$$

Hence it follows that the *limit of an infinitely small quantity α is equal to zero*,

$$\lim \alpha = 0.$$

Sometimes this property of an infinitely small quantity is taken to be its definitive sign, and the following definition is given: *an infinitely small quantity is a variable whose limit is zero.*

One can start from this definition, obtain the property $|\alpha|<\varepsilon$ and then prove all the other properties of infinitesimals as was done in Sec. 46.

Sec. 50. Geometrical Representation of a Number, Variable, and Limit

1°. Any real number a may be represented on a number scale Ox (Fig. 59) by a point whose abscissa is equal to a. Hence, in an analysis, one and the same small letter signifies a number, the

point representing this number, and the abscissa of the point. When we say "the number a", we also mean "the point a", and vice versa.

2°. The variable x may also be represented on a number scale by a point, though in this case by a moving point. Let M on the axis Ox (Fig. 59) move in such a manner that the value

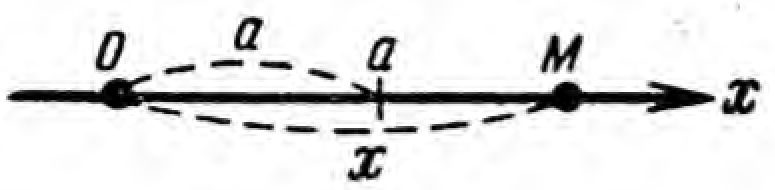

Fig. 59.

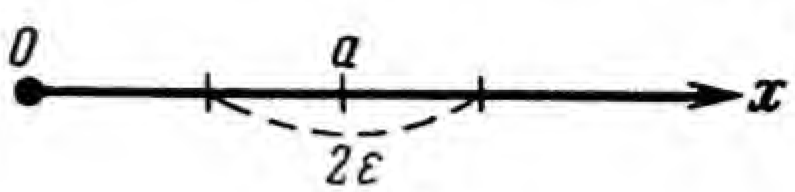

Fig. 60.

of its abscissa is always equal to the numerical value of the variable x. Then this moving point M constitutes the geometrical representation of the variable x.

3°. Let us take a constant point a on the number scale Ox (Fig. 60), and lay off, to the left and right of it, a segment of length ε. We thus obtain a segment of 2ε with midpoint at a.

Definition. *The set of all the points on a segment of length 2ε, of which the midpoint between the ends is a, is termed the neighbourhood 2ε of a.*

The number ε is called the radius of the neighbourhood and a, the centre of the neighbourhood.

Obviously any point a on the axis Ox can have an infinite number of neighbourhoods.

Example. The neighbourhood of the point $a=2$ with radius $\varepsilon=0.1$ consists of all those points on the Ox axis, the abscissas of which satisfy the inequalities

$$2-0.1<x<2+0.1$$

or

$$1.9<x<2.1.$$

Algebraically, the neighbourhood 2ε of the point represents the set of all the values of x satisfying the inequalities

$$a-\varepsilon<x<a+\varepsilon.$$

4°. Let a be the limit of the variable x. In the geometrical sense, $|x-a|$ represents the distance $\overline{Ma}$ (Fig. 61) between the moving point M and the fixed point a. When $|x-a|<\varepsilon$, M lies in the neighbourhood 2ε of a.

It follows from the preceding that the *number a is the limit of the variable x if all the values (points) of x, from and after a certain value lie in the neighbourhood 2ε of the point a, however small the neighbourhood may be taken.*

5°. If x approaches its limit a, being all the time smaller (or larger) than a, then M will approach a from the left (or from the right), and the points representing x values will, from and

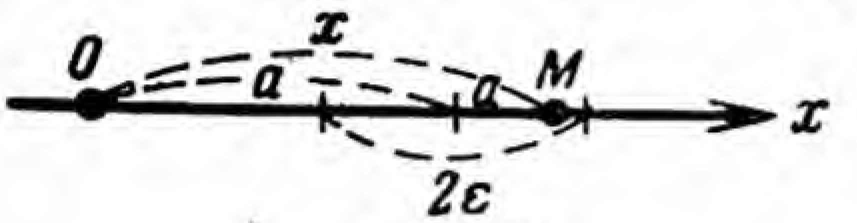

Fig. 61.

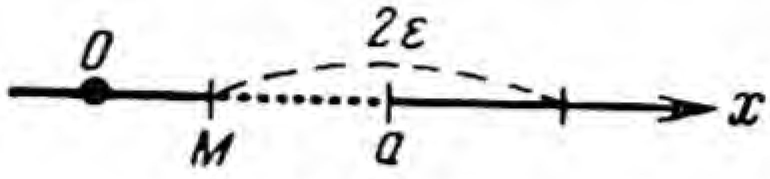

Fig. 62.

after a certain value of x, all accumulate only in the left half (or the right half) of the semi-neighbourhood of the point a (Fig. 62, 63).

If x, as it approaches its limit a, is sometimes larger and sometimes smaller than a, then the point M will oscillate about the

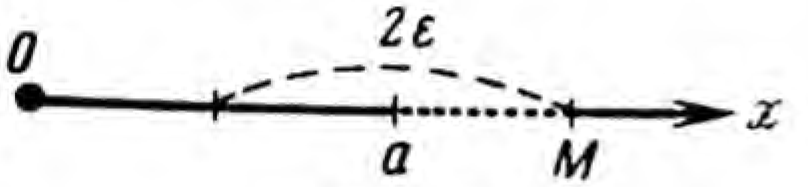

Fig. 63.

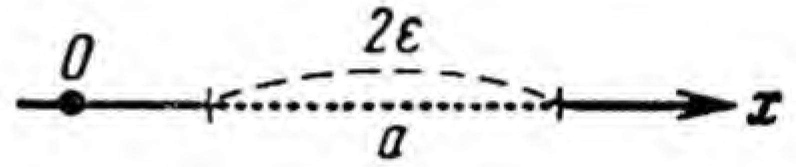

Fig. 64.

point a and the points corresponding to x will — from and after a certain value of x — accumulate in the neighbourhood 2ε of a both to the right of a and to the left of it (Fig. 64).

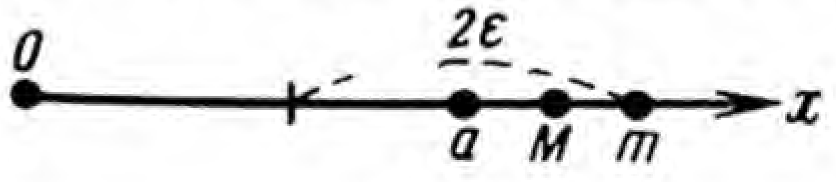

Fig. 65.

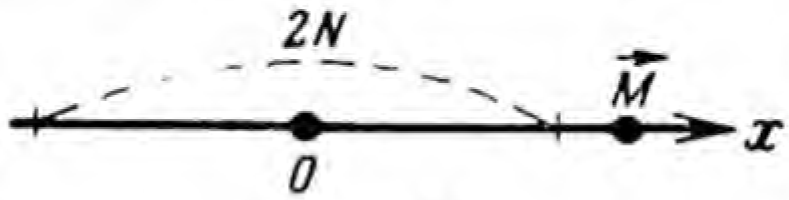

Fig. 66.

6°. The variable x with limit a is a bounded quantity. Indeed, the point M (Fig. 65), after attaining a certain value, always remains within a definite segment of length $|a|+\varepsilon$ and for this reason the following inequality is satisfied:

$$|x| < |a| + \varepsilon.$$

But the converse proposition is not true. For example, $x = \sin\alpha$ is a bounded quantity, since

$$|\sin\alpha| \leqslant 1.$$

However, $\sin\alpha$ has no limit when $\alpha \to \infty$, since the values of $\sin\alpha$ oscillate between -1 and $+1$ and cannot be constantly confined to the neighbourhood 2ε of any particular point.

7°. *An infinitely large quantity has no limit.*

Geometrically this may be represented as follows. Mark out on the axis Ox a neighbourhood of radius N — where N is a positive number — around the point 0 (zero) (Fig. 66). Now, however big N

may be, a time will come, from and after which,

$$|x|>N,$$

and the moving point M, representing an infinitely large quantity, will move beyond the neighbourhood $2N$, and in its further movement will always be outside this neighbourhood.

It is customary to say that "the limit of x is equal to infinity" or "the infinitely large quantity x has infinity as its limit". The notation is $\lim x=\infty$. If x is an infinitely large quantity such that all of its values are positive (or negative) from and after a certain value, it is written:

$$\lim x=+\infty \text{ (or } \lim x=-\infty).$$

No mathematical operations can be carried out on ∞ (infinity); ∞ (infinity) cannot be added, subtracted, multiplied or divided either by itself or by any number, since ∞ is not a number at all.

Sec. 51. Relationship Between a Variable, Its Limit, and an Infinitesimal

The difference between a variable quantity x and its limit a is an infinitesimal quantity, α; for example,

$$x-a=\alpha;$$

because $|x-\alpha|<\varepsilon$. Whence

$$\boxed{x=a+\alpha} \tag{1}$$

i.e., *a variable x having a limit a can be represented as the sum of its limit a and an infinitesimal α.*

Conversely, *if the values of the variable x, from and after the attainment of a certain value, can all be represented as the sum of a number a and an infinitesimal α, then a is the limit of x*, i.e.,

$$\boxed{\text{if } x=a+\alpha, \ \lim x=a} \tag{2}$$

Sec. 52. A Variable Can Have Only One Limit

Theorem. *A variable cannot have more than one limit.*

Proof by reductio ad absurdum. Assume that the variable x has two different limits a and b. Then, from the foregoing,

$$x=a+\alpha \tag{1}$$

and

$$x=b+\beta \tag{2}$$

where α and β are infinitesimals.

Subtracting (2) from (1) we get

$$0=(a-b)+(\alpha-\beta);$$

$$a-b=\beta-\alpha,$$

which is an impossibility because the number $a-b$, not equal to zero, is an infinitely small quantity $\beta-\alpha$ (Sec. 44, 4°).

Corollary. *If a variable has a limit, there can only be one limit.*

Sec. 53. The Limit of an Algebraic Sum

Theorem. *If each term of an algebraic sum of several variables has a limit, then:*

1) *the sum also has a limit,*

2) *this limit of the algebraic sum of several terms is equal to a similar algebraic sum of the limits of these variables.*

Proof. We shall show the theorem to be true for the case of three terms; we assume that variables x, y, z have limits a, b, c, respectively.

Let us write each variable as the sum of its limit and an infinitesimal (Sec. 51):

$$x=a+\alpha,$$

$$y=b+\beta,$$

$$z=c+\gamma,$$

where α, β, γ are infinitesimals.

Adding the first two equations and subtracting the third from this sum, we see that the variable $(x+y-z)$ is equal to the number $(a+b-c)$ added to the infinitesimal $(\alpha+\beta-\gamma)$. Therefore (Sec. 51), *the constant $(a+b-c)$ is the limit of the variable $(x+y-z)$,*

$$\lim(x+y-z)=a+b-c,$$

or

$$\lim(x+y-z)=\lim x+\lim y-\lim z,$$

as required.

Sec. 54. The Limit of a Product

1°. **Theorem.** *If each factor of a product of several variables has a limit, then:*

1) *the product also has a limit,*

2) *the limit of the product of several variables is equal to the product of the limits of the factors.*

Proof. First we shall prove the theorem for a product of two variable factors x and y. We assume that x and y have limits

respectively equal to a and b. Then (by formula 1, Sec. 51)

$$x = a + \alpha$$
$$y = b + \beta$$

where α and β are infinitesimals. Multiplying one equation by the other, we find that the variable (xy) is equal to the number (ab) plus the infinitesimal $(\alpha b + a\beta + \alpha\beta)$, i.e., $xy = ab + (\alpha b + a\beta + \alpha\beta)$.

Hence (Sec. 51) ab is the limit of the variable xy,

$$\lim (x \cdot y) = a \cdot b$$

or

$$\lim (x \cdot y) = \lim x \cdot \lim y,$$

as required.

Proceeding from this proof we shall find the limit of a product of three variables x, y, z, each of which has a limit of its own:

$$\lim (x \cdot y \cdot z) = \lim [(xy) \cdot z] = \lim (xy) \cdot \lim z = \lim x \cdot \lim y \cdot \lim z,$$

since

$$\lim (xy) = \lim x \cdot \lim y.$$

2°. **Corollary 1.** *A constant factor can be taken out of the sign of the limit.*

Indeed, regarding the constant c as a variable having all of its values equal to c, we get

$$\lim c = c,$$
$$\lim (c \cdot x) = \lim c \cdot \lim x = c \cdot \lim x.$$

3°. **Corollary 2.** *If a variable has a limit, then:*

1) *the variable raised to a positive integral power also has a limit,*

2) *the limit of a positive integral power of a variable is equal to the limit of the variable raised to the power in question.*

Let exponent n of the power x^n be a positive integer. Then

$$\lim (x^n) = \overbrace{\lim (x \cdot x \ldots x)}^{n \text{ times}} = \overbrace{\lim x \cdot \lim x \ldots \lim x}^{n \text{ times}} = (\lim x)^n.$$

Sec. 55. The Limit of a Quotient

1°. **Lemma.** *If a positive number a be the limit of a variable x, then, from and after a certain value of x, all subsequent values will be larger than a quantity b, where b is an arbitrary positive number smaller than a.*

Proof. Since $a > 0$, there is an infinity of positive numbers existing between 0 and a. Take one of these numbers b near a,

$$a > b > 0,$$

and construct on the axis Ox a neighbourhood of a with radius equal to $a-b$ (Fig. 67).

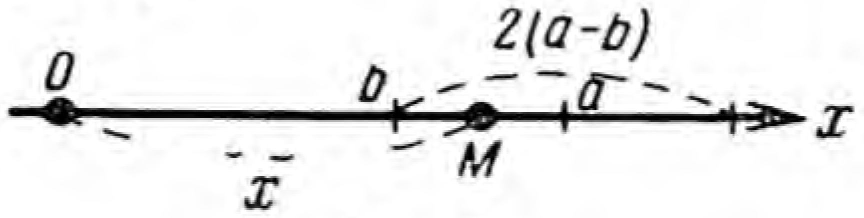

Fig. 67.

By the definition of a limit, the values of x, from and after a certain value, will all lie within this neighbourhood and thus be larger than b.

2° **Theorem 1.** *If a variable x has a limit a not equal to zero, then the inverse quantity $\frac{1}{x}$*

1) *also has a limit,*

2) *which is equal to $\frac{1}{a}$, i. e., to the inverse of the limit of the given variable.*

Proof. Since $a \neq 0$, either $a > 0$ or $a < 0$. Let $a > 0$, i.e., a is positive. We shall show that $\frac{1}{x} - \frac{1}{a}$ is an infinitesimal.

We have

$$\frac{1}{x} - \frac{1}{a} = \frac{a-x}{ax}.$$

The difference $a-x$ is an infinitesimal, for example, α. Then

$$\frac{1}{x} - \frac{1}{a} = \frac{\alpha}{ax} = \frac{1}{a \cdot x} \cdot \alpha.$$

According to the lemma, the values of x, after a certain value, are all greater than b, where b is a positive number smaller than a. Taking only these values of x into account, let us increase the right side of the equation substituting b for x in the denominator. Then

$$\left| \frac{1}{x} - \frac{1}{a} \right| < \frac{1}{ab} \cdot |\alpha|.$$

Since the fraction $\frac{1}{ab}$ is constant and α is an infinitesimal, the whole right side of the inequality is an infinitesimal (Sec. 46, 4°, Corollary 1),

$$\frac{1}{ab} \cdot |\alpha| < \varepsilon.$$

And therefore

$$\left| \frac{1}{x} - \frac{1}{a} \right| < \varepsilon,$$

i. e.,

$$\lim \frac{1}{x} = \frac{1}{a} = \frac{1}{\lim x}.$$

If a is a negative number, then taking the constant factor -1 outside the limit sign will enable us to reduce this case to the one we have considered.

3°. **Corollary.** *If a variable x has a limit $a \neq 0$, then the inverse of the variable, $\frac{1}{x}$, is a bounded quantity.*

4°. **Theorem 2.** *If the dividend and divisor are variables with limits and the limit of the divisor is not equal to zero, then:*

1) *the quotient also has a limit,*

2) *the limit is equal to the quotient of the limits of the dividend and divisor.*

Proof. Let us write the quotient of $\frac{y}{x}$ in the form of a product:

$$\frac{y}{x} = \frac{1}{x} \cdot y.$$

Applying theorems on the limit of a product and the limit of an inverse quantity, we get

$$\lim \frac{y}{x} = \lim \left(\frac{1}{x} \cdot y\right) = \lim \frac{1}{x} \cdot \lim y = \frac{1}{\lim x} \cdot \lim y = \frac{\lim y}{\lim x},$$

as required.

Sec. 56. The Limit of a Rational Algebraic Expression

An algebraic expression is said to be *rational* in x if its value can be obtained from the value of x and other known numbers by means of the following five algebraic operations: addition, subtraction, multiplication, division and raising to a positive integral power.

For example, a polynomial of the form

$$Ax^n + Bx^{n-1} + \ldots + Px + Q,$$

or a fraction of the form

$$\frac{Ax^n + Bx^{n-1} + \ldots + Px + Q}{ax^m + bx^{m-1} + \ldots + kx + l},$$

in which the coefficients are certain numbers and the exponents are positive integers, are rational algebraic expressions in x.

The theorems concerning the limits of an algebraic sum, product, quotient and their corollaries may be generalised thus: *if a variable quantity x has a limit, then:*

1) *any rational algebraic expression in x also has a limit* (assuming that, in the case of a fraction, the denominator has a limit other than zero),

2) *to find the limit of a rational algebraic expression, it is necessary to replace the variable x by its limit and perform the operations indicated in the expression.*

Example. Find the limit of the fraction $\frac{3+2x-x^2}{x^2+2x-3}$ if $x \to 2$, which means that the limit of x is 2. Applying the generalised theorem, we replace x by its limit 2 and get

$$\lim_{x\to 2} \frac{3+2x-x^2}{x^2+2x-3} = \frac{3+2\cdot 2-2^2}{2^2+2\cdot 2-3} = \frac{3}{5}.$$

Sec. 57. The Sign of a Variable and Its Limit

Theorem 1. *The limit of a positive variable (if the latter has a limit) is either positive or zero.*

Proof by reductio ad absurdum. Assume that the limit of x is a negative number a. Between this negative number a and 0 there exists an infinity of negative numbers. Let us take one of these

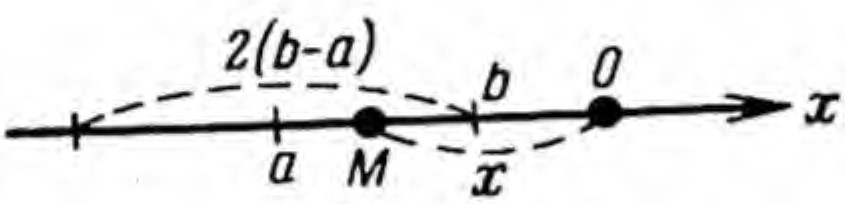

Fig. 68.

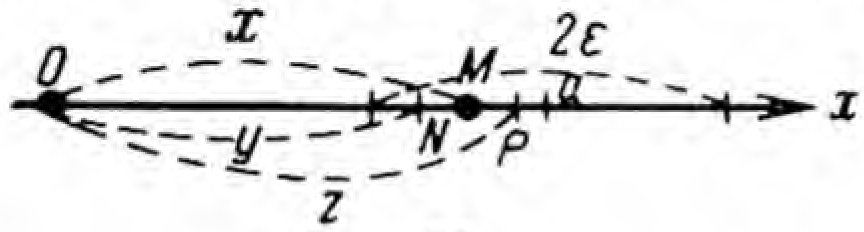

Fig. 69.

numbers b near a and mark out on the axis Ox a neighbourhood of point a with radius $b-a$ (Fig. 68). Any number within this neighbourhood is negative. Since $\lim x = a$, the values of x, from and after a certain value, will all lie in this neighbourhood, i.e., will all be negative. This is impossible since it is given that x has only positive values.

But if the limit of x exists and cannot be negative, it can only be either positive or zero.

Theorem 2. *The limit of a negative variable (if the latter has a limit) is itself either negative or zero.*

Proof is similar to the foregoing.

Sec. 58. Conditions for the Existence of a Limit of a Variable

Theorem 1. *If a variable has values intermediate between those of two other variables having a common limit, then the variable has the same limit.*

Proof. Let

$$y \leqslant x \leqslant z$$

and $\lim y = \lim z = a$. Mark on Ox (Fig. 69) a neighbourhood 2ε of point a. From and after a certain moment, the values of y and z will all belong to this neighbourhood and the values of x will lie within the interval NP, the end points of which are $N(y)$ and

$P(z)$. For this reason, the values of x will belong to the neigh bourhood 2ε of a and, consequently, $\lim x = a$.

Theorem 2. *If, beginning with a certain time, the values of a variable all the time increase (or decrease) but remain smaller (or larger) than a particular number, then the variable has a limit which is smaller (greater) than, or equal to, this number.*

The proof of this theorem lies outside the scope of our course.

Sec. 59. On the Limit of a Quotient of Infinitesimals

1°. *The limit of the quotient of $\frac{\alpha}{x}$ obtained by dividing an infinitesimal α by a variable quantity x having a finite limit a, other than zero, is equal to zero*, because the quotient $\frac{\alpha}{x}$ may be regarded as the product $\left(\frac{1}{x}\right)\cdot\alpha$, i.e., the product of a finite quantity, $\frac{1}{x}$, (Sec. 55, 3°) and an infinitesimal, α, the limit of which is zero (Sec. 46, 4°).

2°. *The limit of a quotient $\frac{x}{\alpha}$ obtained by dividing the variable x with limit a, different from zero, by an infinitesimal, α, is infinity.*

Indeed, the quotient $\frac{x}{\alpha}$ may be treated as the inverse of $\frac{\alpha}{x}$. Since from the foregoing $\frac{\alpha}{x}$ is an infinitesimal, and its inverse $\frac{x}{\alpha}$, is an infinitely large quantity, its limit is infinity (Sec. 50, 7°).

3°. If the dividend α and the divisor β are both infinitesimals, the theorem of the limit of a quotient is not applicable, since its application would give the indeterminate result

$$\lim \frac{\alpha}{\beta} = \frac{\lim \alpha}{\lim \beta} = \frac{0}{0}.$$

It may be noted that the limit of a quotient of two infinitesimals may not always exist.

The solution of some very important technical problems involves the search for the limit of a quotient of infinitesimals (as will be shown in Ch. VI), and the finding of this limit is one of the problems of mathematical analysis.

Sec. 60. Examples in Finding Limits

Let us examine a few cases of finding limits where the theorems of operations on limits cannot be applied.

1°. Find the limit of the fraction

$$\frac{x^2+5x+6}{x^3+x}$$

if $x \to 0$.

Solution. A formal application of the rule in Sec. 56 about the finding of the limit of a rational algebraic expression will suggest the substitution of x by its limit. But then we get

$$\lim_{x\to 0}\frac{x^2+5x+6}{x^3+x}=\frac{0+5\cdot 0+6}{0+0}=\frac{6}{0},$$

which is a meaningless expression. In this case the numerator is a bounded quantity, since its limit is equal to 6, while the denominator is an infinitesimal, since its limit is zero. Hence (by Sec. 59, 2°) the limit of the given fraction is infinity.

2°. Find the limit of the fraction $\frac{x^2-3x+2}{2x^2-5x+2}$ if $x\to 2$.

Solution. Replacing x by its limit, we get

$$\lim_{x\to 2}\frac{x^2-3x+2}{2x^2-5x+2}=\frac{4-6+2}{8-10+2}=\frac{0}{0}.$$

Since zero cannot be divided by zero, the example presents a case where the theorems of operations on limits are not applicable. In such cases, an attempt is made to convert the expression into another one (identical with the first), to which the theorems of operations on limits are applicable, and then to find the limit.

In the given instance, factorise the numerator and denominator and reduce the fraction. The roots of the numerator are 2 and 1, of the denominator, 2 and $\frac{1}{2}$:

$$\lim_{x\to 2}\frac{x^2-3x+2}{2x^2-5x+2}=\lim_{x\to 2}\frac{(x-2)\cdot(x-1)}{2\,(x-2)\left(x-\frac{1}{2}\right)}=$$

$$=\lim_{x\to 2}\frac{(x-2)\,(x-1)}{(x-2)\,(2x-1)}=\lim_{x\to 2}\frac{x-1}{2x-1}=\frac{2-1}{4-1}=\frac{1}{3}.$$

3°. Find the limit of $\frac{x-5}{x+8}$ when $x\to\infty$.

Solution. When x increases indefinitely, both the numerator $x-5$ and the denominator $x+8$ are infinitely large quantities.

Hence:

$$\lim_{x\to\infty}\frac{x-5}{x+8}=\frac{\infty}{\infty}.$$

We again have an instance of the non-applicability of the theorems of operations on limits, since ∞ is not subject to any mathematical operation. Let us transform the given fraction by dividing the numerator and denominator by x termwise:

$$\lim_{x\to\infty}\frac{x-5}{x+8}=\lim_{x\to\infty}\frac{1-\frac{5}{x}}{1+\frac{8}{x}}=\frac{1-0}{1+0}=1.$$

Here, 0 is the limit of the infinitely small quantities $\frac{5}{x}$ and $\frac{8}{x}$.

4°. Find the limit of the sum $\frac{1}{x-a}+\frac{x-3a}{x^2-a^2}$ if $x\to a$.

Solution. If $x \to a$, then $x-a \to 0$, $x-3a \to -2a$, $x^2-a^2 \to 0$, and each fraction is an infinitely large quantity. Again the theorems of operations on limits cannot be applied since they lead to the expression $\infty+\infty$.

Transform the given expression by reducing the fractions to a common denominator and adding them:

$$\lim_{x\to a}\left(\frac{1}{x-a}+\frac{x-3a}{x^2-a^2}\right)=\lim_{x\to a}\frac{x+a+x-3a}{(x-a)(x+a)}=$$

$$=\lim_{x\to a}\frac{2(x-a)}{(x-a)(x+a)}=\lim_{x\to a}\frac{2}{x+a}=\frac{2}{a+a}=\frac{1}{a}.$$

CHAPTER V

FUNCTION AND ITS CONTINUITY

Sec. 61. Argument and Function

1°. Definition. *The quantity y is said to be a function of x if to every permissible numerical value of the argument x there corresponds a single definite numerical value of y.*

"Permissible" means such values of the independent variable for which the function has real numbers for its values.

Examples. 1. For the function $y=\sqrt{x}$, positive numbers and zero are the permissible values of x, since negative values of x give imaginary values to the function.

2. For $y=\arcsin x$, permissible values of the argument are all numbers x satisfying the conditions $-1\leqslant x\leqslant +1$ since only for these values of x are values of y possible.

3. The perimeter P of a regular n-gon inscribed in a circle of radius R is given by the formula:

$$P=2Rn\sin\frac{180^\circ}{n}.$$

The perimeter P is a function of the number of sides n: only positive integers greater than 2 are permissible values of n.

4. For the function $y\frac{x}{x^2-1}$, permissible values of x are all real numbers with the exception of -1 and $+1$. When $x=\pm 1$ the denominator x^2-1 becomes zero and the fraction $\frac{x}{x^2-1}$ becomes meaningless, since division by zero is impossible.

2°. *The set (aggregate) of all permissible values of the argument forms the domain of definition of the function.*

In particular, the domain of definition of a function may be an *open* or *closed interval* or *both*.

The set of all real numbers x satisfying the condition $a\leqslant x\leqslant b$, where $a<b$, *is called a closed interval.*

The closed interval $a\leqslant x\leqslant b$ in abbreviated form is denoted as $[a, b]$. Geometrically, the closed interval $a\leqslant x\leqslant b$ is a set of

all the points belonging to the segment ab of the number scale Ox (Fig. 70), including the end points of the segment, a and b.

The set of all real numbers x satisfying the two inequalities $a < x < b$ is called an open interval.

The open interval $a < x < b$ is denoted as (a, b).

Geometrically, the open interval $a < x < b$ is a set of all points on the number scale Ox (Fig. 70) lying between the points $x = a$

Fig. 70.

and $x = b$, the points $x = a$ and $x = b$ themselves not forming part of the interval (a, b).

Note that the set of all real numbers x is denoted by the inequalities

$$-\infty < x < +\infty.$$

3°. **Examples.** 1. The domain of definition of the function $y = \arcsin x$ is the interval $-1 \leqslant x \leqslant +1$.

2. The domain of definition of the function $P = 2Rn \sin \frac{180°}{n}$, where P is the perimeter of a regular n-gon inscribed in a circle, is given by the set of all integral numbers n in the interval $2 < n < +\infty$.

3. Three intervals constitute the domain of definition of $y = \frac{x}{x^2-1}$: $-\infty < x < -1$, $-1 < x < +1$ and $+1 < x < +\infty$.

We will also say: the function $y = \frac{x}{x^2-1}$ is defined in the intervals $-\infty < x < -1$, $-1 < x < +1$, and $+1 < x < +\infty$.

4. The definition of a function does not require that y must necessarily change with a change in the value of the argument x. It is sufficient if y has a definite value for every permissible value of x.

Example. Let y be the reading of an electric meter and x the time. y is a function of x since for every instant of time x the meter indicates the amount of electric power consumed, y. The function y remains unchanged during those intervals of time when no power is consumed.

We may conceivably have a function

$$y = c$$

where c is a constant quantity denoting that the value of y is one and the same for all values of c. The graph of such a function is a straight line parallel to the x-axis or coinciding with it if $c = 0$.

5°. The definition of function is usually associated with Dirichlet. However, a few years before Dirichlet defined function, the definition was given in an article by the great Russian mathematician N. I. Lobachevsky "On the Disappearance of Trigonometrical Lines", published in 1834.

Sec. 62. General Designation of a Function

1°. That y is a definite function of x is indicated by the equation

$$y = f(x)$$

which is read "y is equal to f of x" or "y is a function of x", the letter f here signifying the dependence of y on x, i.e., the rule under which, for every permissible value of the argument x, a corresponding value of the function y can be definitely determined.

In accordance with the notation $y = f(x)$, the fact that the length l of a rod is a function of the temperature t is written

$$l = f(t),$$

and that the volume V of a gas is a function of the pressure p (temperature being constant):

$$V = \varphi(p).$$

Here the dependence is signified by the symbol φ, and not by f, to underline that the relationship between V and p is different from the relationship between l and t, e.g.,

$$f(t) = l_0(1 + \alpha t),$$

while

$$\varphi(p) = \frac{c}{p}.$$

Thus the symbols $f(x)$, $\varphi(x)$, $F(x)$, etc., signify different functions of one and the same argument x, while the symbols $f(x)$, $f(t)$, $f(z)$ signify one and the same function of different arguments.

Examples:

1) $f(x) = \frac{c}{x}$; $\qquad f(t) = \frac{c}{t}$;

2) $f(x) = \sin x$; $\qquad f(z) = \sin z$.

2°. The numerical value of a function $f(x)$ for any particular value of the argument x is signified by writing the numerical value of x in the symbol $f(x)$. For example, the symbols $f(2)$ and $f(a)$ signify numbers respectively equal to the value of $f(x)$ when $x = 2$ and $x = a$.

If $f(x)=x^2$, then $f(2)=2^2=4$, and $f(a)=a^2$.

If $f(x)=\sin x$, $f\left(\frac{\pi}{6}\right)=\sin\frac{\pi}{6}=\frac{1}{2}$.

3°. The symbol $f(x)$ can also help us write the properties of a function. For example, the property of logarithms that the log of a product = sum of the logs of its factors, $\log_a(x\cdot y)=\log_a x+\log_a y$ can be written with f designating the logarithm:

$$f(x\cdot y)=f(x)+f(y).$$

Or, the property of sines:

$$\sin(x+y)=\sin x\cdot\cos y+\cos x\cdot\sin y,$$

can be stated in functional notation (signifying sin by f and cos by φ) thus:

$$f(x+y)=f(x)\cdot\varphi(y)+\varphi(x)\cdot f(y).$$

Sec. 63. Graphical and Analytical Representation of a Function

Of the different methods of representing a function (in words, in tabular form, graphically, by formula, etc.), the graphical and analytical methods are of particular importance for our purposes.

1°. If in the plane of a system of rectangular coordinates xOy a curve be drawn such that on any straight line parallel to the

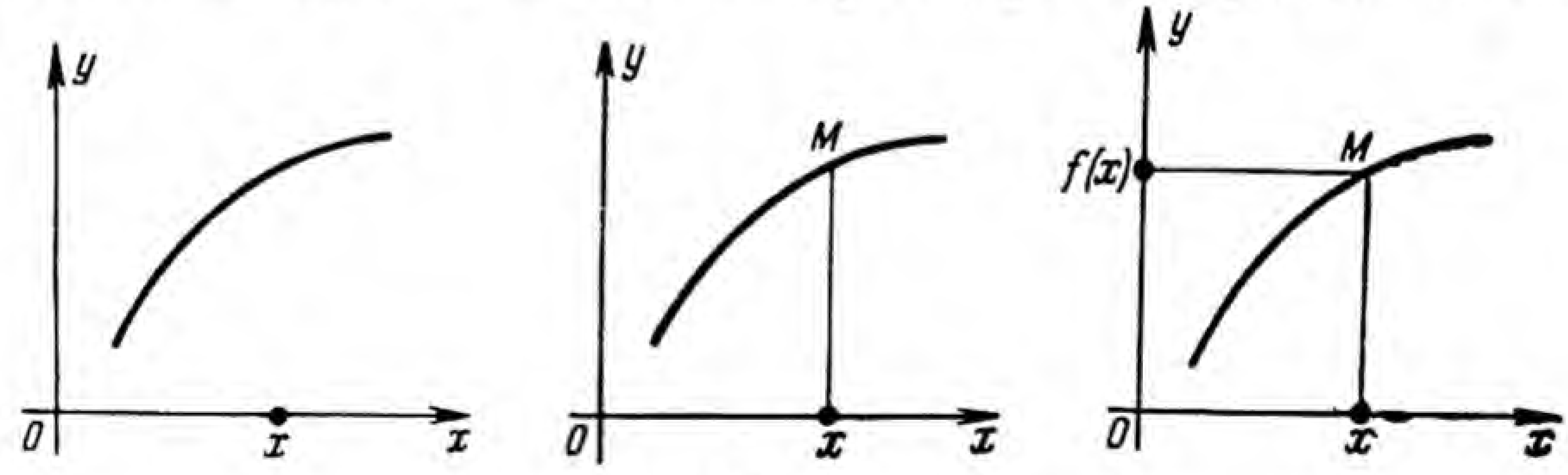

Fig. 71.

y-axis there lies not more than one point of the curve, then the curve signifies some function $y=f(x)$, and the value of $f(x)$ for any given value of x is determined in this case geometrically as follows:

1) the point x is marked on the x-axis (Fig. 71, *a*);

2) a straight line parallel to Oy is drawn through x until it intersects the given line at M (Fig. 71, *b*);

3) a parallel to Ox is drawn from M and intersects the axis Oy at the point $f(x)$ (Fig. 71, *c*).

As a result of these three operations, point x is *mapped* into point $f(x)$; $f(x)$ — the value of the function — corresponds to the number x; the number $f(x)$ is the ordinate of the point $f(x)$.

The domain of definition of the function $y=f(x)$ represented graphically is the set of abscissas of all the points of the given line.

2°. If the functional relation is expressed by a formula, we say that the function is defined analytically. For example, in the formula for the area of a circle,

$$s=\pi r^2,$$

the area s is analytically shown to be a function of the radius r.

For the analytical determination of a single function it is sometimes necessary to employ a number of different formulas. For example, the reading of the electric meter at the commencement of the day was 20 kwh. Then during the day the following consumption of electricity took place: from 5 to 8 o'clock, 5 lamps of 60 w each, and from 17 to 24 o'clock a 100-watt radio and five 60-watt lamps. Denoting the meter reading by y and the time by x, we obtain equations of the dependence of y upon x:

$$y=\begin{cases} 20, & \text{when } 0\leqslant x\leqslant 5; \\ 20+0.3\,(x-5), & \text{when } 5\leqslant x\leqslant 8; \\ 20.9, & \text{when } 8\leqslant x\leqslant 17; \\ 20.9+0.4\,(x-17), & \text{when } 17\leqslant x\leqslant 24. \end{cases}$$

3°. A function is said to be *explicit* if the formula defining the function indicates directly the mathematical operations to be performed with the argument in order to determine the function y. Otherwise, the function is said to be *implicit*.

Examples: 1) $y=ax^2$, $y=f(x)$ are explicit functions;

2) $y^2-x=0$, $F(x, y)=0$ are implicit functions.

4°. Functions studied in algebra and trigonometry:

1) $y=x^n$, where n is a given rational number;

2) the exponential function $y=a^x$, where $a>0$, $a\neq 1$, $-\infty<$ $<x<+\infty$;

3) the logarithmic function $y=\log_a x$, where $a>0$, $a\neq 1$, $0<x<+\infty$;

4) the trigonometric functions $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\operatorname{cosec} x$;

5) the inverse trigonometric functions: $\arcsin x$, $\arccos x$, $\arctan x$, etc., are called *basic elementary functions*.

The function $y=x^q$, where q is an irrational number, and the above-mentioned exponential, logarithmic, trigonometric and inverse trigonometric functions are called *elementary transcendental functions*.

Those functions are also called elementary which are obtained by doing a number of simple mathematical operations of addition,

subtraction, multiplication or division on basic functions and by employing the signs of the basic elementary functions.

For example, $f(x)=\sqrt{\dfrac{\arccos x-\sqrt{\sin x}}{\log(1+3^{\tan x})}}$

is an elementary function.

Sec. 64. Graph of a Function

1°. If for every value of x within the domain of definition of the function $y=f(x)$ a point is constructed in a plane, the abscissa of the point being equal to the given value of x, and the ordinate, to the corresponding value of $f(x)$, then the locus of $M[x, f(x)]$ thus obtained is called the graph of the given function.

The construction of the graph of a function is based on investigation of the variation of the function. The methods of such investigation will be indicated later on. For the present, when constructing the graph of a given function, we shall use methods of algebra to find the peculiarities of its variation.

2°. Let us draw the graph of the function $y=x^3$. This function is defined for all values of x from $-\infty$ to $+\infty$; for positive values of x the function is positive and for negative values, negative. The construction of every single point of the graph corresponding to values of x in the interval $-\infty<x<+\infty$ is obviously impossible practically. Let us assign several values to x and calculate the corresponding values of y from the formula $y=x^3$.

For $x=$	-2	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$\to+\infty$
$y=$	-8	$-3\frac{3}{8}$	-1	$-\frac{1}{8}$	0	$\frac{1}{8}$	1	$3\frac{3}{8}$	8	$\to+\infty$
Point	M	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	

Taking x as the abscissa and y as the ordinate, construct the points M, M_1, M_2, ... (Fig. 72) according to their values in the table, and draw a smooth curve through the points M_1, M_2, M_3..., etc. This curve is the graph of the function $y=x^3$.

The abscissa of each point of this curve is equal to the value of the argument x, and the ordinate is equal to the corresponding value of the function $y=x^3$.

3°. The graph of the function $y=|x|$ is a broken line (Fig. 73) consisting of the bisectors of the quadrantal angles of the second and the first quadrant.

4°. The function $y=\frac{|x|}{x}$ is defined for all values of x except $x=0$. In the interval $-\infty<x<0$, $y=-1$, and in the interval $0<x<+\infty$, $y=+1$. The graph (Fig. 74) consists of two

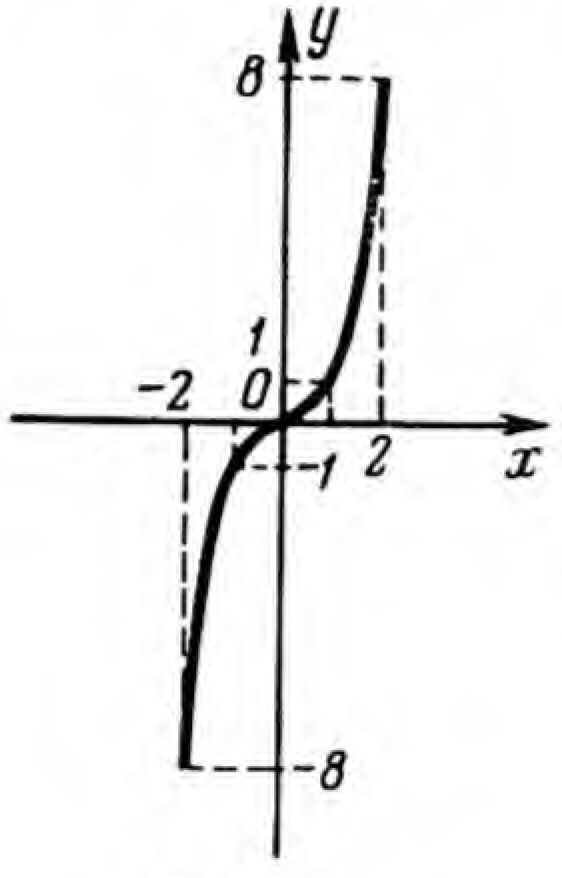

Fig. 72.

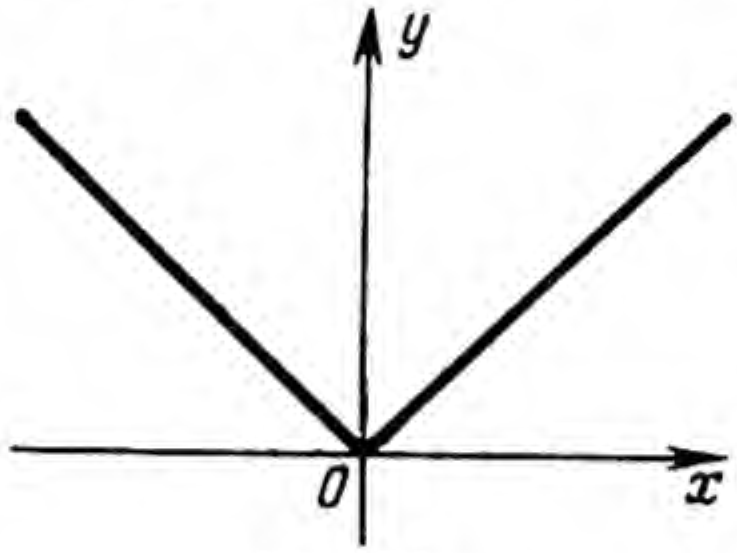

Fig. 73.

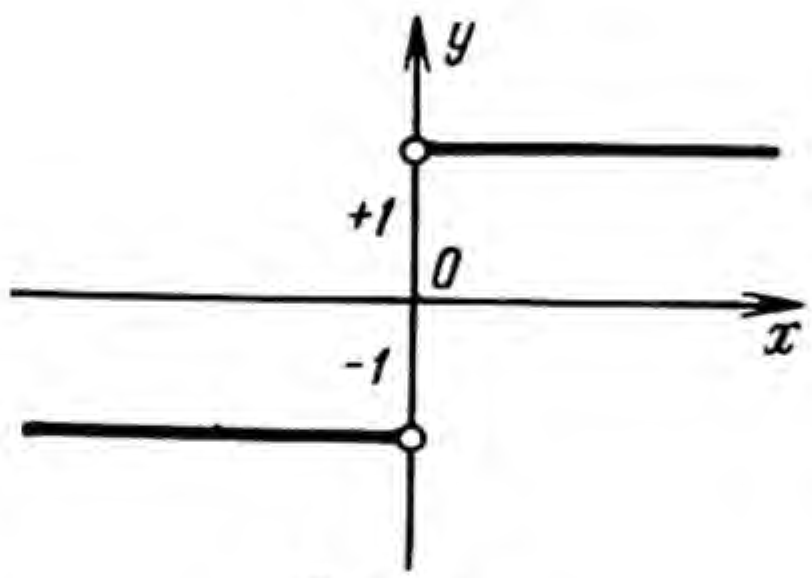

Fig. 74.

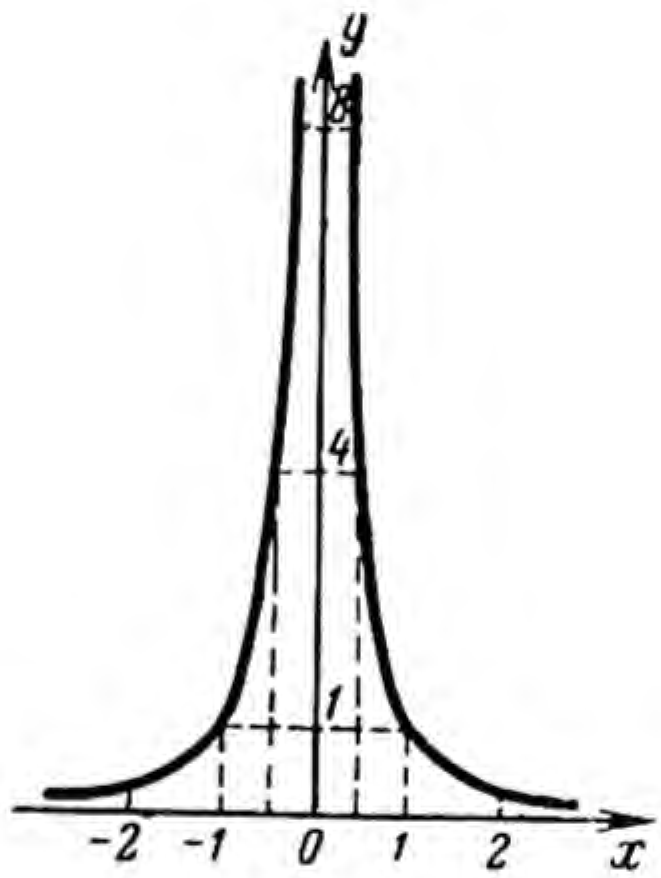

Fig. 75.

half-lines $y=-1$ and $y=+1$. The first does not have the last point and the second ($y=1$) does not have the first point. They represent, as it were, two parts of a single straight line out of which the point $x=0$ has been extracted, and the parts thus obtained are displaced parallel to the x-axis at a distance equal to 1: one below the x-axis and the other above it.

5°. The function $y=\frac{1}{x^2}$ is defined for all values of x, except $x=0$, and is positive. The graph lies entirely above the x-axis and consists of two branches, one to the left of the y-axis and the other, to the right of it. When x approaches zero either from the

side of positive values or from the side of negative values, y increases without bound, the graph rises without limit higher and higher in the form of a spire with the y-axis as the axis of symmetry. When the absolute value of x increases without bound, y tends to zero and both branches of the curve approach, without bound, the x-axis.

Let us construct the graph (Fig. 75) using the following points:

If $x=$	± 2	± 1	$\pm \frac{1}{2}$	$\pm \frac{1}{4}$	$\to 0$
$y=$	$+\frac{1}{4}$	$+1$	$+4$	$+16$	$\to +\infty$

Sec. 65. Increment of the Argument and Function

1°. Let us take two arbitrary values of the argument x in the domain of definition of the function $y=f(x)$: the first we shall call the initial value and the second, the altered value.

The initial value of x is assumed constant during the course of the following discussion, and its corresponding point A (Fig. 76) on the x-axis fixed. It is customary to denote the altered value of x by $x+\Delta x$. Its corresponding point in Fig. 76 is P.

Δx is the quantity by which the argument varies in its transition from the first to the second value and is called the increment of the argument.

Δx is equal to the difference between the second and the first value of the argument.

Examples. 1. Given: two values of x,

$$3.0 \text{ and } 3.1; \quad \Delta x = 3.1 - 3.0 = 0.1.$$

2. Given: two values of x, 5 and 4.8;

$$\Delta x = 4.8 - 5 = -0.2.$$

3. Given: two values of x, -1 and -0.98;

$$\Delta x = -0.98 - (-1) = 0.02.$$

2°. To the values x and $x+\Delta x$ of the argument there correspond definite values of the function: the initial, y, and the altered, $y+\Delta y$.

Δy is the quantity by which the value of the function y changes in the variation of the argument by Δx, and is called the increment of the function.

Δy is equal to the difference between the second and the first value of the function.

Let us plot the points $M(x, y)$ and $M_1(x+\Delta x, y+\Delta y)$ of the graph of the function $y=f(x)$ (Fig. 76). $\Delta y = NN_1 = ON_1 - ON$.

Geometrically, the increment of the function Δy is the difference between the ordinates of the points of the graph of the function corresponding to the altered and initial values of the argument.

The increment of the function Δy can be positive as well as negative. When Δy is positive, the segment $NN_1 = \Delta y$ on the axis

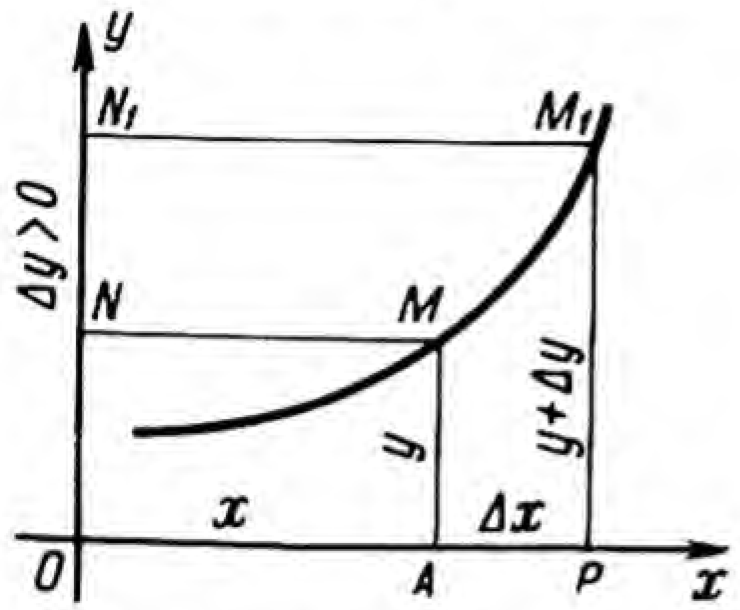

Fig. 76.

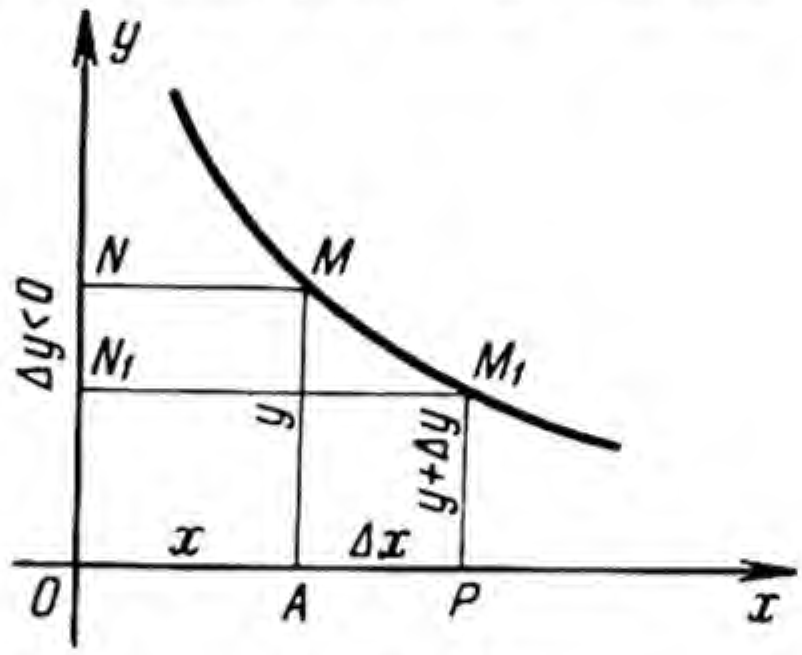

Fig 77.

of ordinates (Fig. 76) is above the fixed point N, and when it is negative, below it (Fig. 77).

3°. **Examples.** 1. By how much is the area y of a square altered when the length of its side x changes by Δx?

Solution. The area of a square

$$y = x^2.$$

If the length of the side of the square is $x+\Delta x$, then its area will be $y+\Delta y$, and

$$y+\Delta y = (x+\Delta x)^2.$$

Subtracting the initial value of the area from the altered value, we get

$$\Delta y = (x+\Delta x)^2 - x^2.$$

Removing brackets, we find that the area has changed to

$$\Delta y = 2x\cdot\Delta x + \Delta x^2.$$

If, for example, the side increases from 3 to 3.1 metres, the area of the square will increase by

$$\Delta y = 2x\cdot\Delta x + \Delta x^2 = 2\cdot3\cdot0.1 + 0.1^2 = 0.61 \text{ (m}^2\text{)},$$

since $x=3$ m and $\Delta x = 0.1$ m.

2. Find the increment Δy of the function $y=\frac{1}{x}$ corresponding to an arbitrary increment Δx of the argument x.

Solution. When the argument has the value x, the function has the value

$$y = \frac{1}{x},$$

and when the argument$=x+\Delta x$, the function is

$$y+\Delta y=\frac{1}{x+\Delta x}.$$

Subtracting the first equation from the second, we obtain the increment of the function:

$$\Delta y=\frac{1}{x+\Delta x}-\frac{1}{x}=\frac{x-x-\Delta x}{x(x+\Delta x)}=-\frac{\Delta x}{x(x+\Delta x)}.$$

For example, increasing the argument x from 4 to 4.5 gives the function an increment

$$\Delta y=-\frac{0.5}{4\cdot 4.5}=-\frac{1}{36}$$

(since $x=4$, $x+\Delta x=4.5$ and $\Delta x=0.5$), i.e., the function decreases by $\frac{1}{36}$.

4°. Let us find the expression for the increment of the function $f(x)$ due to a change of Δx in the argument x.

The initial value of the function is

$$y=f(x).$$

Its altered value is

$$y+\Delta y=f(x+\Delta x).$$

By subtraction we find the increment of the function:

$$\boxed{\Delta y=f(x+\Delta x)-f(x)}$$

Sec. 66. The Limit of a Function at a Finite Point

1°. *If the value of the argument x of a function $y=f(x)$ tends towards the number c (according to what law is immaterial), and $x\neq c$, and we can find a time, after which the absolute value of the difference between the value of the function $f(x)$ and the number A becomes less than a given positive number ε, however small:*

$$|f(x)-A|<\varepsilon,$$

then the number A is the limit of the function $f(x)$ at the point c.

Note that when speaking of the limit of a function $f(x)$ at c it is always assumed that $x\neq c$, i.e., that

$$0<|x-c|.$$

2°. Example. Let $f(x)=x^3$. We shall show that $\lim\limits_{x\to 2} x^3=8$.

It is known that

$$|x^3-8|=|x^2+2x+4|\cdot|x-2|.$$

Let us examine how x approaches to 2 within some definite neighbourhood of the point 2, say, within the neighbourhood of a radius equal to 1, i.e.,

$$2-1<x<2+1 \text{ or } 1<x<3.$$

For any value of x in this neighbourhood

$$7<|x^2+2x+4|<19$$

and

$$|x^3-8|<19\cdot|x-2|. \tag{I}$$

Let us take an arbitrary small positive number ε. From equation (I) it follows that

$$|x^3-8|<\varepsilon \quad \text{if} \quad |x-2|<\frac{\varepsilon^*}{19}=\delta.$$

Since the number 2 is the limit of x, a value of x will be found, from and after which the inequality $|x-2|<\delta$ and, consequently, the inequality $|x^3-8|<\varepsilon$ will be fulfilled. This means that $\lim\limits_{x\to 2} x^3=8$.

3°. The concept of the limit of a function given in 1° can now be defined exactly.

Definition. *The number A is the limit of the function $f(x)$ at the point c when:*

1) *x tends to c (no matter by what law) and $0<|x-c|$;*

2) *for any given small positive number ε there exists a positive number δ such that absolute value of the difference between $f(x)$ and A is less than ε if the absolute value of the difference between x and c is less than δ, i.e.,*

$$\boxed{|f(x)-A|<\varepsilon, \quad \text{if} \quad |x-c|<\delta} \tag{II}$$

4°. Geometrically, fulfilment of the inequality

$$|f(x)-A|<\varepsilon$$

after fulfilment of the inequality

$$|x-c|<\delta$$

signifies that, if on the axis Oy we construct a neighbourhood of the point A of arbitrarily small radius ε (Fig. 78), then a radius δ can be found for such a neighbourhood of c on the axis Ox that all values of the argument x (with the exception of $x=c$) belonging to the neighbourhood 2δ of c, determine the values of the function belonging to the neighbourhood 2ε of the point A.

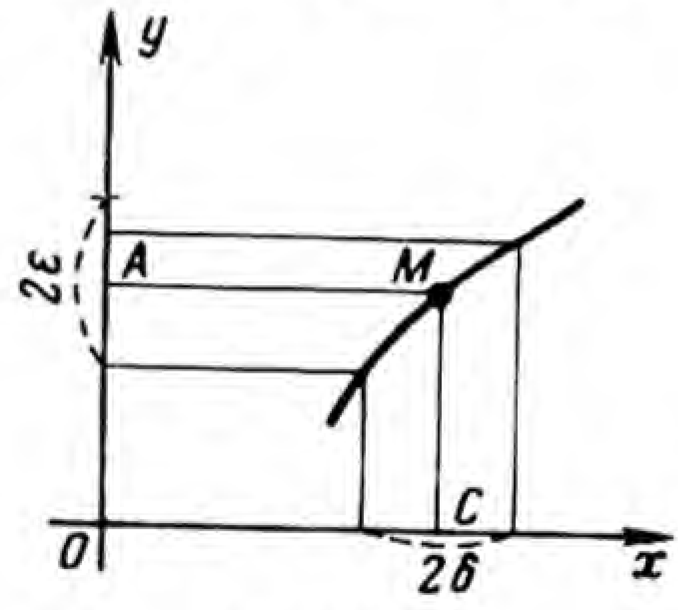

Fig. 78.

5°. *It may happen that when $x\to c$ the function will be an infinitely large quantity.* For example, the function

$$y=\frac{1}{x^2}$$

(Fig. 75) is an infinitely large quantity when $x\to 0$.

---

* Here, $\frac{\varepsilon}{19}$ must be less than the radius of the neighbourhood, 1.

In this case we say that at the point $x=c$ the function has an infinite limit.

6°. **Definition.** $+\infty$ (*or* $-\infty$) *is the infinite limit of a function $f(x)$ at the point c, when*

1) *x tends to c, and $0<|x-c|$,*

2) *for any large positive number N there exists a positive number δ such that*

$$f(x)>N \quad (\text{or} \quad f(x)<-N) \quad \text{if} \quad |x-c|<\delta.$$

In this example:

$$\frac{1}{x^2}>N \quad \text{if} \quad x^2<\frac{1}{N}$$

or if $|x-0|<\delta=\frac{1}{\sqrt{N}}$.

Sec. 67. The Limit of a Function when $x \to \infty$

1°. Some functions have a finite limit A when the argument x increases (or decreases) without bound.

Example 1. $f(x)=\frac{2x+1}{x}$ has 2 for its limit when $x \to \pm\infty$ (Fig. 79). Indeed,

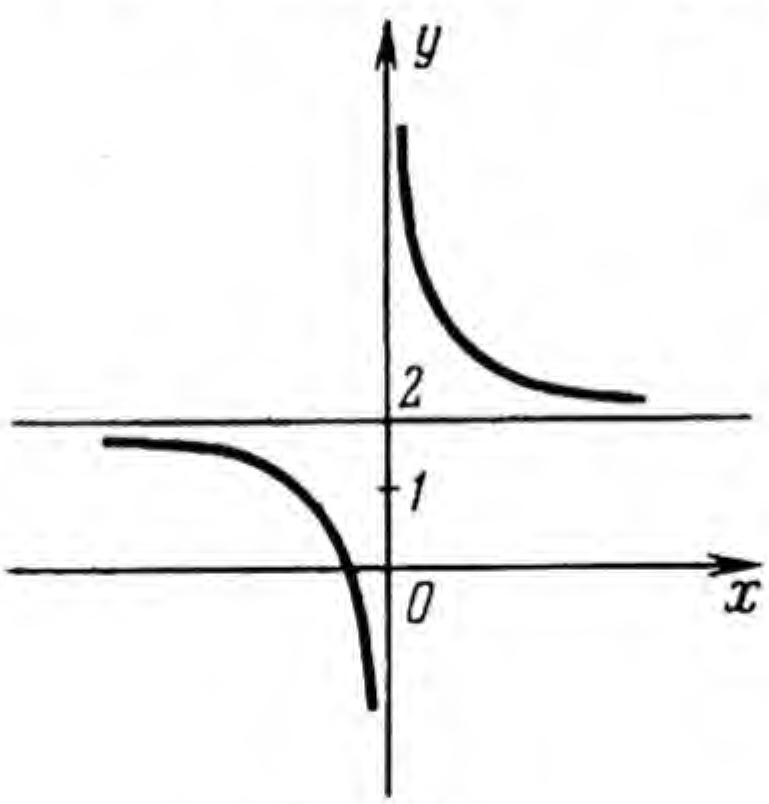

Fig. 79.

$$f(x)=\frac{2x+1}{x}=2+\frac{1}{x}.$$

Whence

$$f(x)-2=\frac{1}{x}.$$

Taking an arbitrarily small positive number ε, we have

$$|f(x)-2|<\varepsilon, \quad \text{if} \quad |x|>\frac{1}{\varepsilon}.$$

Hence,

$$\lim_{x\to(\pm\infty)} f(x)=2.$$

Example 2. $\lim\limits_{x\to(-\infty)} 10^x=0$. Indeed since $10^x>0$, for any definite value of x

$$10^x-0<\varepsilon \quad \text{if} \quad x<\log\varepsilon.$$

Example 3. $\lim\limits_{x\to(+\infty)} 10^x=+\infty$. Indeed, we may take any positive number N, however large, and we will find that

$$10^x>N \quad \text{if} \quad x>\log N.$$

2°. **Definition.** *The number A is the limit of the function $f(x)$ as $x \to \infty$, if for a given small positive number ε there exists a positive number N such that*

$$|f(x)-A| < \varepsilon \quad \text{if} \quad x > N \quad (\text{or } x < -N).$$

Sec. 68. Some Observations

1°. By definition (Sec. 66) the number A is the limit of the function $f(x)$ at the point c when $x \longrightarrow c$ *by any law whatsoever.* Sometimes it happens that the limit of the function $f(x)$ at c

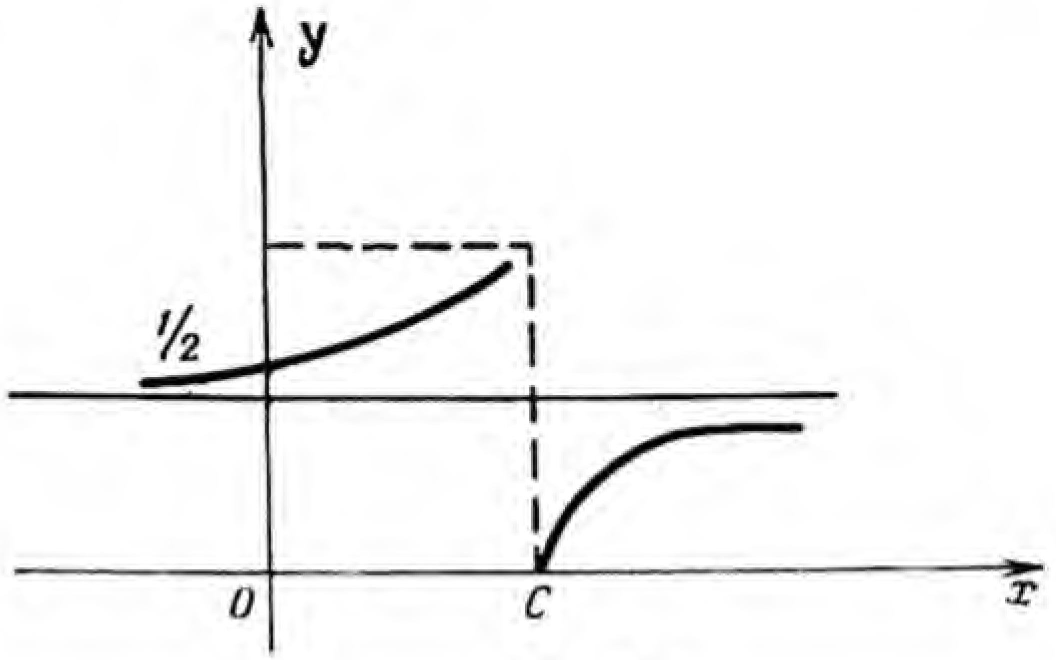

Fig. 80.

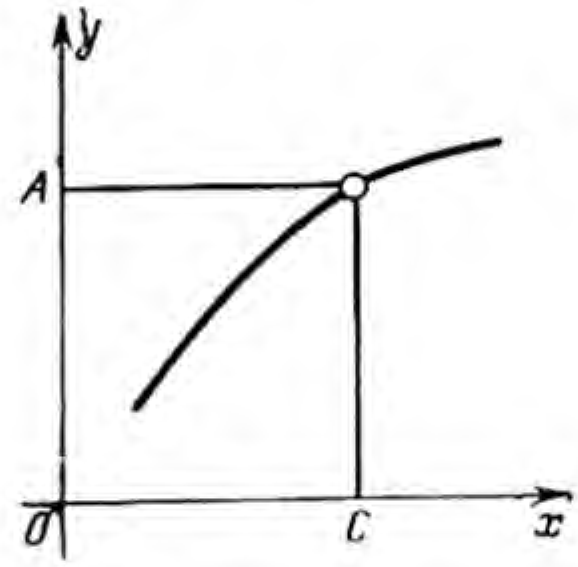

Fig. 81.

differs when x tends to c from the left (i.e., remains less than c) and when x tends to c from the right (i.e., is greater than c). For example, the function

$$y = f(x) = \frac{1}{1+10^{\frac{1}{x-c}}}, \text{ where } c > 0,$$

has the limit 1 "on the left" and the limit 0 "on the right" (Fig. 80).

Indeed, let us denote the value of the fraction $\frac{1}{x-c}$ by z,

$$z = \frac{1}{x-c}.$$

If x approaches c from the left $(x < c)$, then the difference $x-c$ approaches zero, remaining negative all the time. And z in this case will tend to $-\infty$. Since $\lim\limits_{z \to (-\infty)} 10^z = 0$,

$$\lim_{\substack{x \to c \\ x < c}} f(x) = \frac{1}{1+0} = 1.$$

If x approaches c from the right $(x > c)$, then the difference $x-c$ will approach zero, remaining positive all the time. And z

in this case will tend to $+\infty$. Since $\lim\limits_{x\to(+\infty)} 10^x = +\infty$, the fraction

$$\frac{1}{1+10^x}$$

has 0 for its limit:

$$\lim_{\substack{x\to c\\ x>c}} f(x) = 0.$$

When the limits of a function at the point c "from the left" and "from the right" differ, the function cannot, in the ordinary sense of the term, be said to have a limit at c.

2°. The limit of a function $f(x)$ at c—number A—should not be confused with the number $f(c)$, which is the value of the

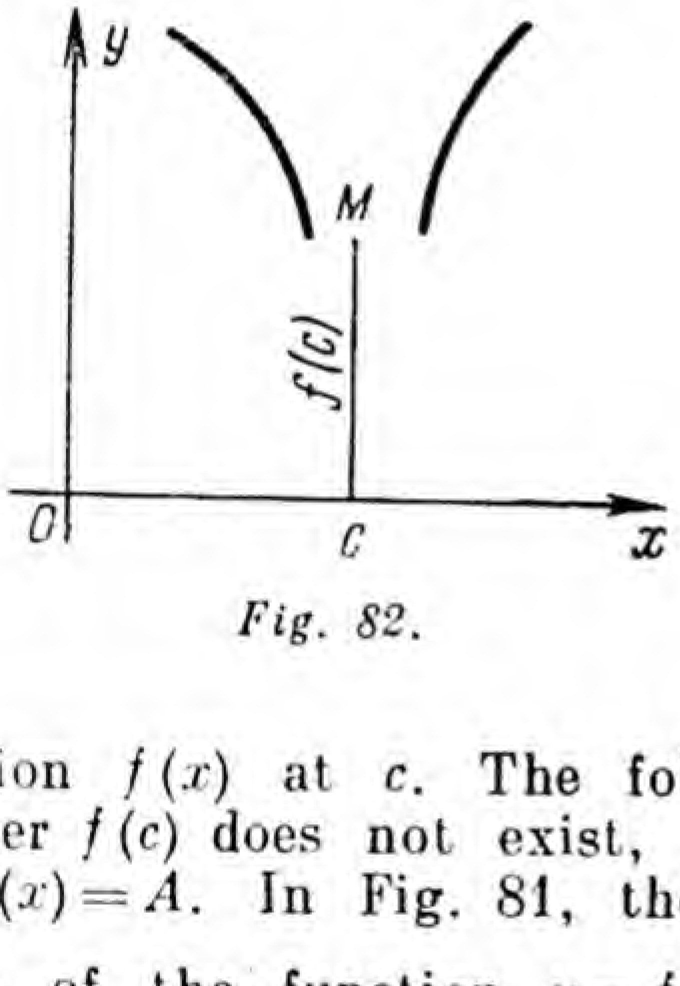

Fig. 82.

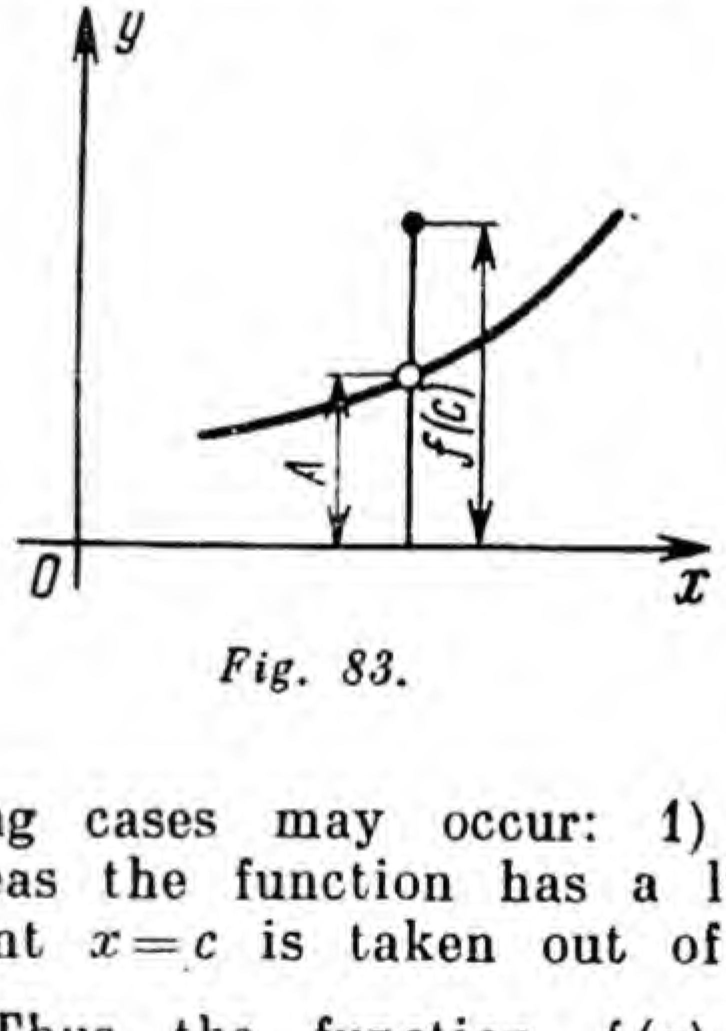

Fig. 83.

function $f(x)$ at c. The following cases may occur: 1) the number $f(c)$ does not exist, whereas the function has a limit $\lim\limits_{x\to c} f(x) = A$. In Fig. 81, the point $x = c$ is taken out of the graph of the function $y = f(x)$. Thus the function $f(x)$ has no value at $x = c$, but the limit $\lim\limits_{x\to c} f(x)$ exists, and on the graph it is equal to OA;

2) the function has a value at the point $x = c$ (Fig. 82), $f(c) = CM$, but the function has no limit at $x \to c$;

3) the value of the function at the point c, $f(c)$ and the limit of the function at that point, A, both exist, but $A \neq f(c)$. (Fig. 83).

Sec. 69. Continuity of a Function

1°. We shall make clear by a concrete example the meaning of the concept "continuity of a function at a point".

Let it be required to compute the value of the function $y = x^3$ when $x = \sqrt{2}$. Now $\sqrt{2}$ is the limiting value of a series of its

approximate values:

$$x = 1.4, \quad 1.41, \quad 1.414, \ldots$$

or $x = 1.5, \quad 1.42, \quad 1.415, \ldots$

as the degree of accuracy of calculation increases:

	x	x^3
First approximation . . .	1.4 and 1.5	2.7 and 3.4
Second approximation . .	1.41 and 1.42	2.80 and 2.87
Third approximation . . .	1.414 and 1.415	2.827 and 2.833

In Fig. 84, the values of the function $y = x^3$ are represented to a first approximation by points M_1 and M'_1, to a second approximation, by points M_2 and M'_2, to a third approximation by M_3 and M'_3, ..., and through each point are drawn straight lines parallel to Ox and Oy. The straight lines drawn through M_2 and M'_2 parallel to Ox are more closely spaced than those drawn through M_1 and M'_1; and, similarly, those drawn through M_3 and M'_3 are still more closely spaced than those drawn through M_2 and M'_2, and so on.

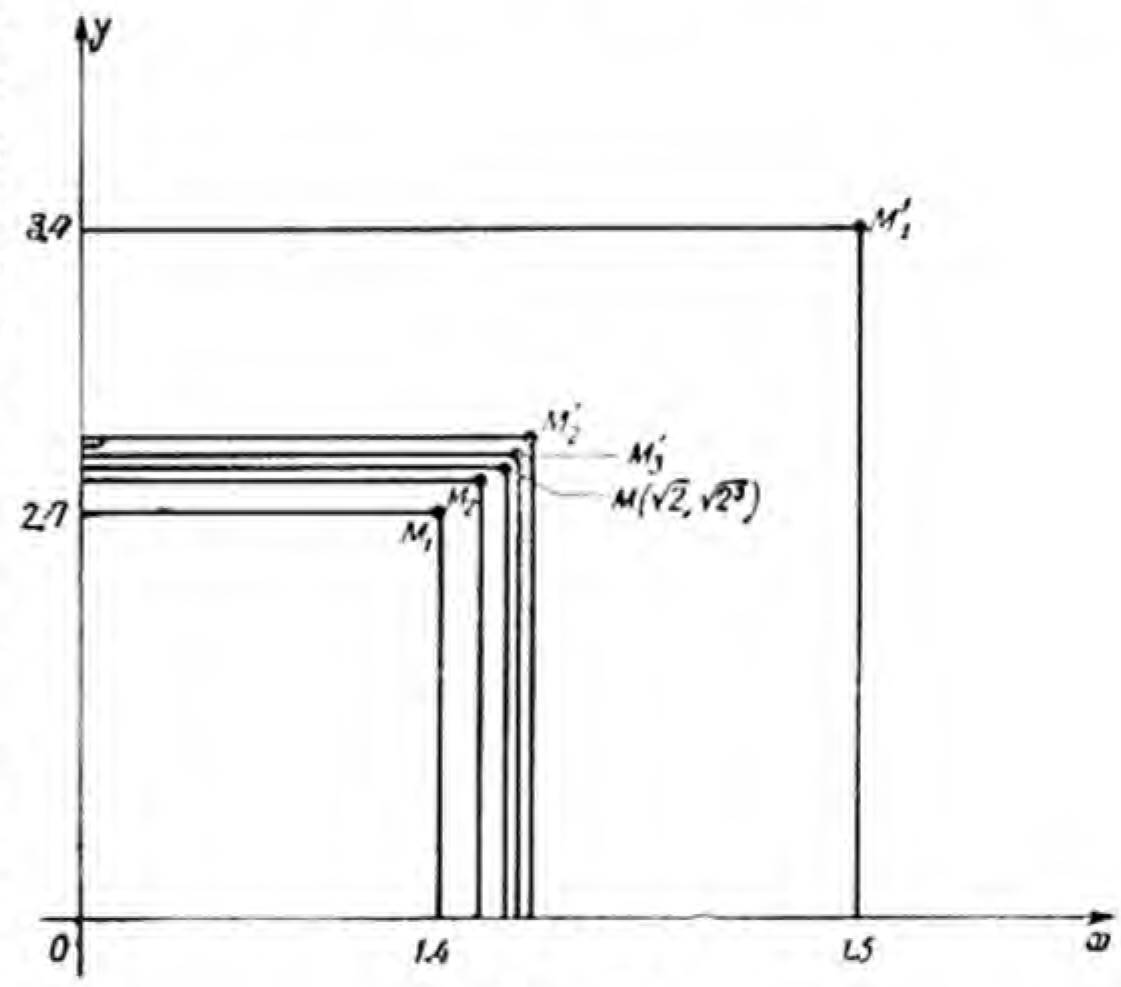

Fig. 84.

The distance between the straight lines parallel to Ox can be made as small as we please by sufficiently bringing together the straight lines parallel to the axis Oy, and the point M will always lie in the rectangle thus formed. Consequently, *the given function $y = x^3$ has the value $(\sqrt{2})^3$ at the point $x = \sqrt{2}$ and has a limit also equal to $(\sqrt{2})^3$.* This property of the function is decisive in defining the continuity of a function at a point.

2°. **Definition.** 1. *The function $f(x)$ is said to be continuous at the point $x = c$ if the limit of the function at c is equal to its value at this point, i.e., if*

$$\boxed{\lim_{x \to c} f(x) = f(c)} \qquad \text{(III)}$$

Thus the condition for continuity of a function $f(x)$ at a point $x=c$ is that:

a) the value of the function at $x=c$ be some definite number $f(c)$;

b) the limit of the function $f(x)$, when x tends to c on the left as well as on the right, be one and the same definite number, $\lim\limits_{x\to c} f(x)$ *;

c) the numbers $\lim\limits_{x\to c} f(x)$ and $f(c)$ be equal.

2. *Every point where the function is continuous is called a point of continuity of the function.*

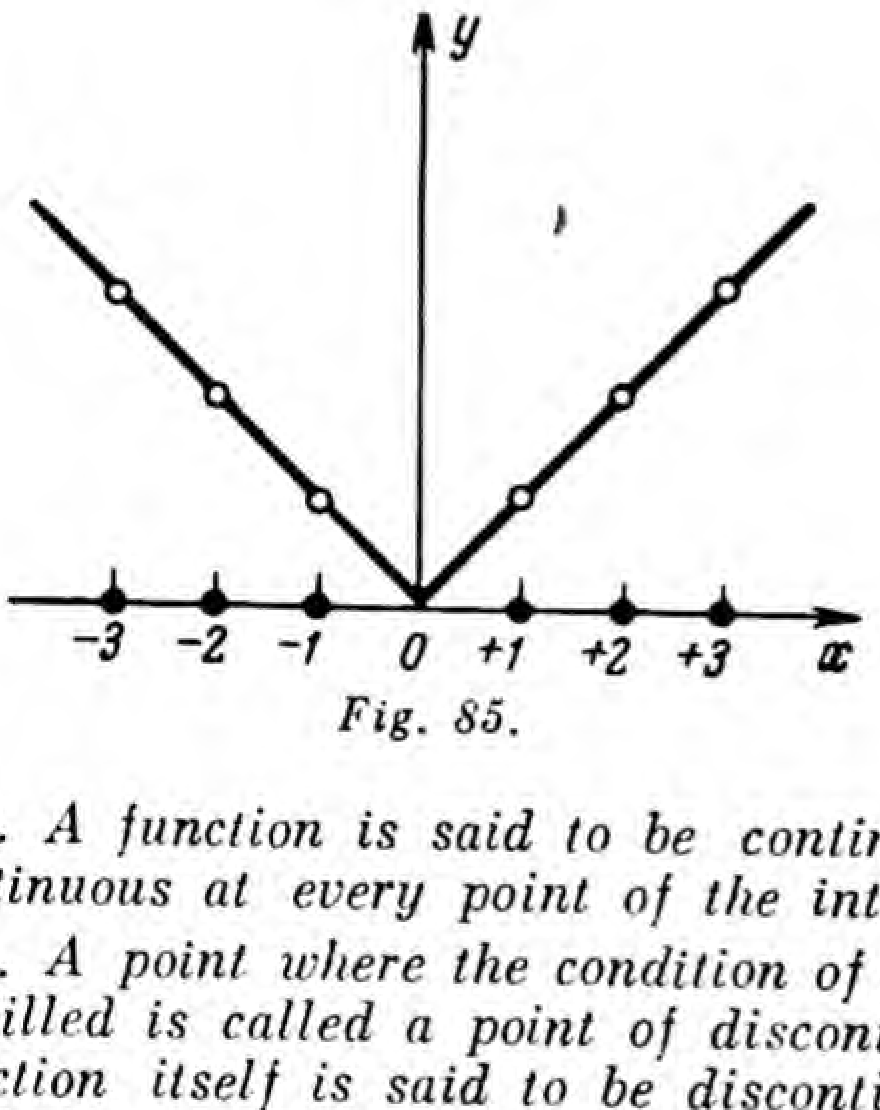

Fig. 85.

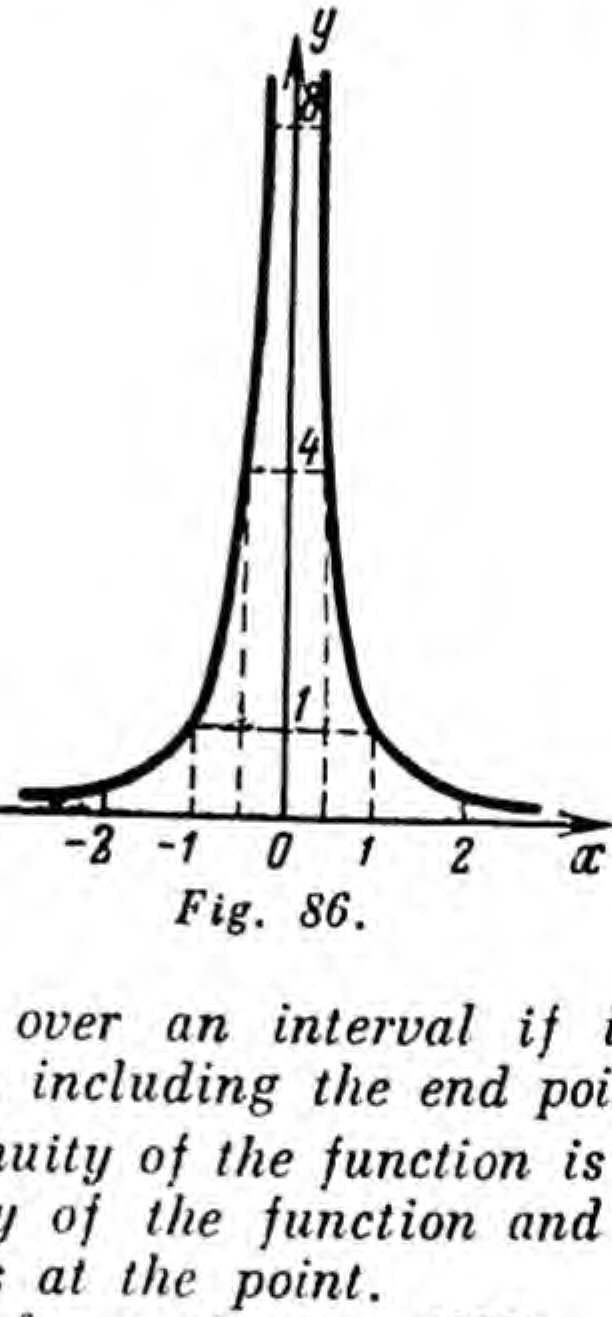

Fig. 86.

3. *A function is said to be continuous over an interval if it is continuous at every point of the interval, including the end points.*

4. *A point where the condition of continuity of the function is not fulfilled is called a point of discontinuity of the function and the function itself is said to be discontinuous at the point.*

3°. Non-fulfilment of the condition of continuity (III) may consist, for example, in the following.

1. The limit $f(x)$ at the point c is not the same on the left as on the right. For example: a) the function $f(x)=\dfrac{1}{1+10^{\frac{1}{x-c}}}$, where $c>0$ (Fig. 80), at $x=c$ has a limit 1 on the left and a limit 0 on the right (Sec. 68), hence $x=c$ is the point of discontinuity; b) the function $f(x)=\dfrac{|x|}{x}$ (Fig. 74) at $x=0$ has a limit -1 on the left and a limit $+1$ on the right, $x=0$ is the point of discontinuity of the function.

* The function $f(x)$ is one-sided if the point $x=c$ serves as the boundary point of the domain of the function.

2. The limit $f(x)$ at c is not equal to the value of the function when $x=c$. For example, the function $f(x)=|x|$ if x is any real but non-integral number, and 0 if x is an integer; it has one and the same limit 3, 2, 1, 1, 2, 3 on the left as well as on the right at points -3, -2, -1, 1, 2, 3, respectively, but none of these limits is equal to the value of the function at these points, which value is zero. The graph of this function (Fig. 85) is a broken line consisting of the bisectors of the coordinate angles in the second and the first quadrant, the points having integral values of the abscissa "taken out" of the bisectors. The values of the function at these points are situated on the x-axis.

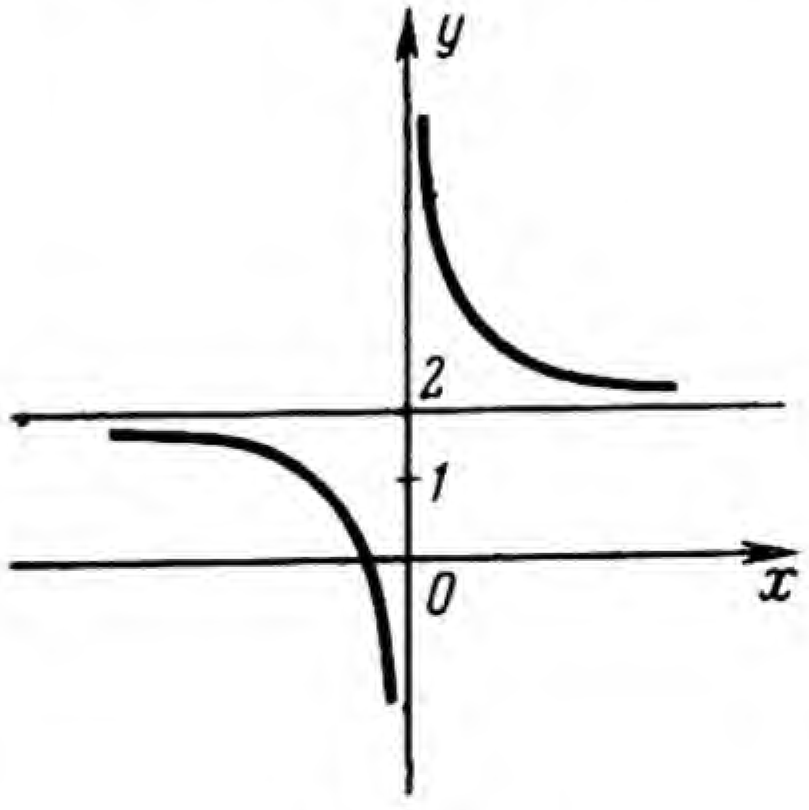

Fig. 87.

When $x=-3, -2, -1, 1, 2, 3$, etc., the function is discontinuous.

3. The limit of the function $f(x)$ at c is infinite. For example, the function $y=\frac{1}{x^2}$ (Fig. 86) at the point $x=0$ has an infinite, not a finite, limit (Sec. 66, 5°); $x=0$ is the point of discontinuity.

The function $y=\frac{2x+1}{x}=2+\frac{1}{x}$ (Sec. 67) at $x=0$ has the limit $-\infty$ on the left and the limit $+\infty$ on the right (Fig. 87), and no point on the graph represents the point $x=0$. When $x=0$ the function is discontinuous.

4°. The foregoing instances of discontinuities in functions occur in engineering practice. For example, girders used in construction work often carry a load like this: to the left of a given cross-section the load is distributed evenly lengthwise and has one value, to the right it is also distributed evenly but has quite a different value. Thus at the given cross-section of the girder there is a jump in the linear distribution of load along the girder. The law of this distribution of load corresponds to a discontinuous function, and the jump in linear load distribution corresponds to the point of discontinuity of the function. Loads concentrated at individual points of a girder may correspond to isolated points of the graph of distribution of load along the length of the girder.

5°. Equality (III) may be replaced by two inequalities:

$$|f(x)-f(c)|<\varepsilon \text{ if } |x-c|<\delta$$

and the continuity of the function of the point is represented geometrically as follows (Fig. 88). Construct on the axis Oy a neighbourhood of the point $f(c)$ of any small radius 2ε. If the function $y=f(x)$ is continuous at c, then on the axis Ox one can find around the point c a neighbourhood of radius δ such that all its points are mapped into the neighbourhood 2ε of

the point $f(c)$. In other words, if we draw two strips: one of width 2ε bounded by two straight lines parallel to the axis Ox, $y=f(c)+\varepsilon$ and $y=f(c)-\varepsilon$, and the other of width 2δ bounded by two straight lines parallel to the axis Oy ($x=c+\delta$ and $x=c-\delta$), then all the points of the graph of the function lying in the second strip belong also to the first strip.

This condition is not satisfied in the case of a discontinuity in the function; namely, a strip of some particular width 2ε bounded by straight lines parallel to the axis Ox will be found such that a point of the graph lying

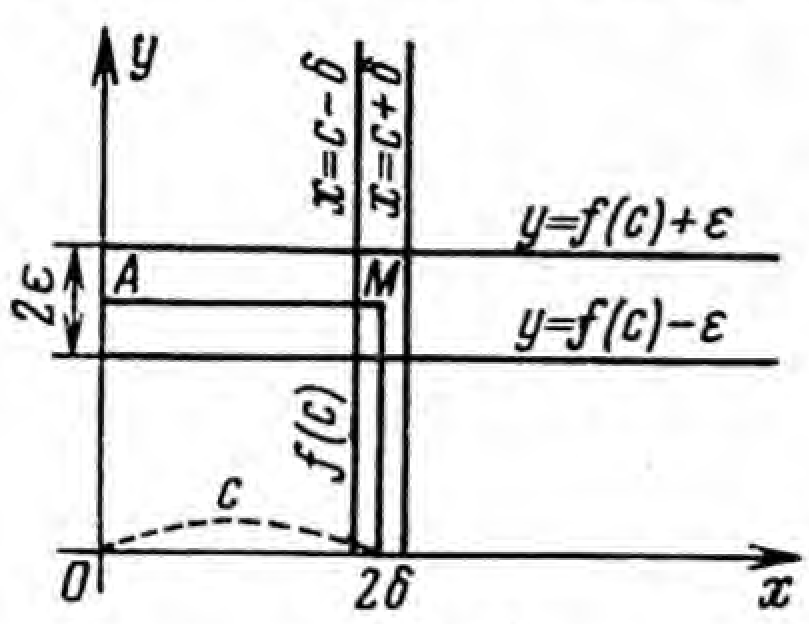

Fig. 88.

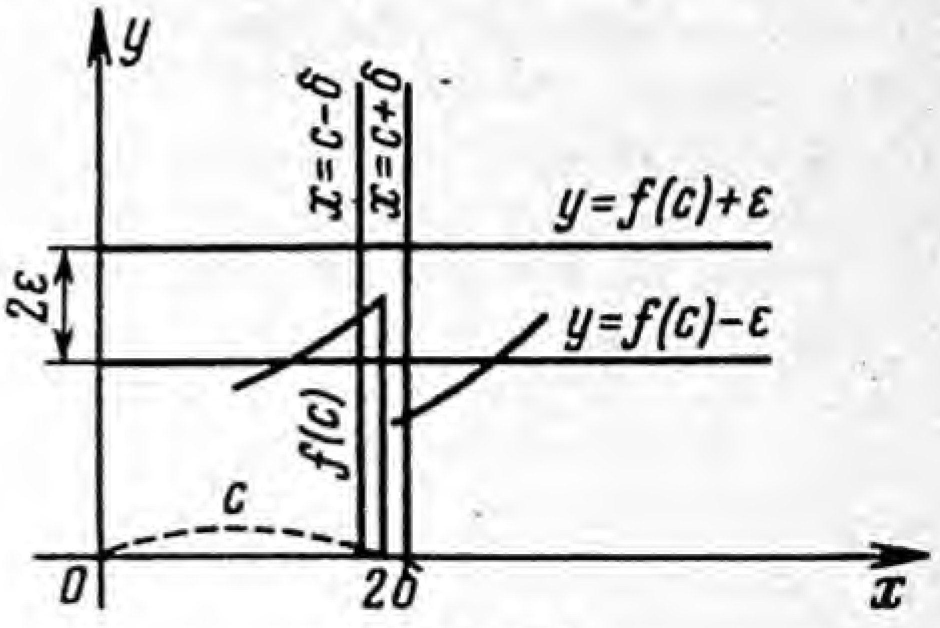

Fig. 89.

in some other strip of width 2δ bounded by two straight lines parallel to the axis Oy will lie outside the first strip, however small we may choose the width 2δ of the second strip (Fig. 89).

Sec. 70. Another Expression for the Condition of Continuity of a Function

From the condition of continuity (III) it follows that

$$\lim_{x\to c}(x-c)=0, \tag{1}$$

$$\lim_{x\to c}[f(x)-f(c)]=0 \tag{2}$$

(the validity of these equations can be verified by finding limits by the theorem of the limit of an algebraic sum).

But the difference $x-c$ expresses the increment Δx of the argument at c,

$$x-c=\Delta x.$$

Note that

$$\lim \Delta x=0.$$

The difference $f(x)-f(c)$ expresses the increment Δy of the function due to the increment of the argument at c,

$$f(x)-f(c)=\Delta y.$$

That x tends to c is equivalent to Δx tending to zero. Therefore equation (2) may be written in the form:

$$\boxed{\lim_{\Delta x\to 0}\Delta y=0} \tag{IV}$$

i. e., for a function continuous at a point the increment of the function is an infinitely small quantity when the increment of the argument at this point becomes an infinitely small quantity.

Sec. 71. Testing a Function for Continuity

Example. We shall prove that $\sin x$ is a continuous function for any value of x. This can be done by showing that for $\sin x$, at any point $x=c$,

$$\lim_{\Delta x\to 0} \Delta y=0.$$

Now, denoting the values of the sine respectively by y and $y+\Delta y$ when its arc is equal to c and $c+\Delta x$, we have

$$y=\sin c \text{ and } y+\Delta y=\sin (c+\Delta x).$$

Subtracting the first equation from the second, we get

$$\Delta y=\sin (c+\Delta x)-\sin c,$$

or, by trigonometry:

$$\sin \alpha-\sin \beta=2 \cos \frac{\alpha+\beta}{2} \cdot \sin \frac{\alpha-\beta}{2}.$$

$$\Delta y=2 \cos \left(c+\frac{\Delta x}{2}\right) \cdot \sin \frac{\Delta x}{2}.$$

In absolute values:

$$|\Delta y|=2\cdot\left|\cos \left(c+\frac{\Delta x}{2}\right)\right|\cdot\left|\sin \frac{\Delta x}{2}\right|.$$

$\left|\cos\left(c+\frac{\Delta x}{2}\right)\right|$ can never exceed unity; taking this expression to be equal to 1, we get $\left|\sin\frac{\Delta x}{2}\right| \leqslant \frac{|\Delta x|}{2}$ (Sec. 44, 3°). Hence $\Delta y \leqslant 2\cdot 1\cdot\frac{|\Delta x|}{2}$, i. e., $|\Delta y| \leqslant |\Delta x|$. Since it is given that $|\Delta x|<\varepsilon$ so also, $|\Delta y|<\varepsilon$ or

$$\lim_{\Delta x\to 0} \Delta y=0,$$

and, consequently, $\sin x$ is a function continuous for all values of x.

Sec. 72. The Properties of Functions Continuous at a Point

1°. *Finding the limit of a continuous function may be replaced by finding the value of the function of the limit of the argument.*

Indeed, since $c=\lim\limits_{x\to c} x$,

equation (III) may be written as

$$\boxed{\lim_{x\to c} f(x)=f(\lim_{x\to c} x)} \qquad \text{(V)}$$

This property is sometimes expressed briefly thus: *the limit sign of a continuous function can be referred to the argument.*

2°. It may be proved (though we shall not here reproduce the proof) that elementary transcendental functions (Sec. 63, 4°) are

continuous at any point in the domain of their definition. Hence formula (V) can be applied to them to obtain

$$\lim \sin x = \sin (\lim x), \qquad \lim \arctan x = \arctan (\lim x),$$

$$\lim \log x = \log (\lim x).$$

3°. *The function $f(x)$ that is constant in the neighbourhood of the point c is continuous at that point.*

Proof. Given that $f(x)$ is equal to some constant quantity, say, k, in the neighbourhood of $x=c$: $f(x)=k$. Therefore

$$\lim_{x \to c} f(x) = k.$$

But since, likewise,

$$f(c) = k,$$

we have

$$\lim_{x \to c} f(x) = f(c)$$

consequently, the function is continuous at the point c, as required.

4°. *The sum, difference, product or quotient of two continuous functions at the point $x=c$ is itself a continuous function provided, in the case of the quotient, the divisor (denominator) is not zero.*

Proof. Let $f(x)$ and $\varphi(x)$ be two functions continuous at the point $x=c$, i. e., let $\lim\limits_{x \to c} f(x) = f(c)$ and $\lim\limits_{x \to c} \varphi(x) = \varphi(c)$. Then, on the basis of the theorems of limits (Secs. 51, 52, 53), we get

1. $\lim\limits_{x \to c} [f(x) \pm \varphi(x)] = \lim\limits_{x \to c} f(x) \pm \lim\limits_{x \to c} \varphi(x) = f(c) \pm \varphi(c)$;
2. $\lim\limits_{x \to c} [f(x) \cdot \varphi(x)] = \lim\limits_{x \to c} f(x) \cdot \lim\limits_{x \to c} \varphi(x) = f(c) \cdot \varphi(c)$;
3. $\lim\limits_{x \to c} \dfrac{f(x)}{\varphi(x)} = \dfrac{\lim\limits_{x \to c} f(x)}{\lim\limits_{x \to c} \varphi(x)} = \dfrac{f(c)}{\varphi(c)}$, if $\varphi(c) \neq 0$.

5°. It follows from the foregoing that the polynomial

$$Ax^n + Bx^{n-1} + \ldots + Px + Q$$

and the fraction

$$\frac{Ax^n + Bx^{n-1} + \ldots + Px + Q}{ax^m + bx^{m-1} + \ldots + kx + l}$$

(the fraction assumed to be irreducible) *are functions continuous for all values of the argument except for those which, in the case of the fraction, reduce the denominator to zero.*

CHAPTER VI

DERIVATIVE FUNCTION

Sec. 73. Linear Function, Its Rate of Change

1°. A function expressed by the equation

$$y = kx + b$$

is called *linear*, and its graph is a straight line.

Let us find the increment Δy of a linear function due to an increment Δx in the argument:

$$\Delta y = k(x + \Delta x) + b - (kx + b) \quad \text{or} \quad \boxed{\Delta y = k \cdot \Delta x} \qquad \text{(I)}$$

This equation expresses the basic property of linear functions: the *increment of a linear function is directly proportional to the increment of its argument.*

Geometrically this property is demonstrated by the fact that for one particular increment Δx of the abscissas, the increments Δy of the ordinates are equal for all points of the straight line $y = kx + b$ (Fig. 90).

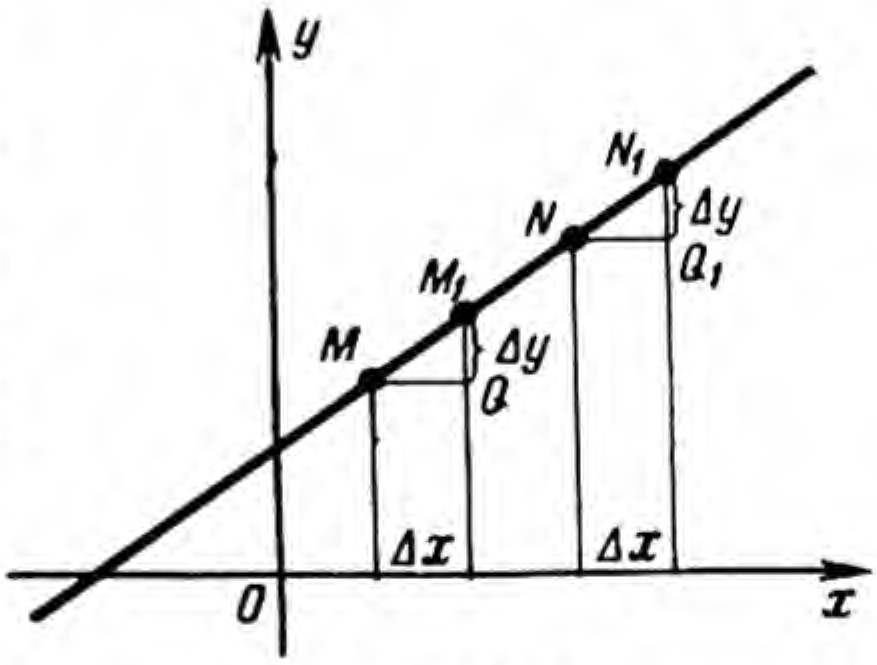

Fig. 90.

2°. No other function has this property.

Let us prove that *any function $y = f(x)$, the increment Δy of which is directly proportional to the increment Δx of its argument, is a linear function.*

Given that $y = f(x)$; $\Delta y = k \cdot \Delta x$. It is required to prove that $f(x) = kx + b$.

Proof. Taking the initial value of the argument to be zero and the altered value x, we obtain for the increment of the argument: $\Delta x = x - 0 = x$. Under these conditions the initial value of the function $y = f(x)$ is a constant $f(0)$, which may be denoted by the letter b,

$$f(0) = b.$$

The altered value of the function is $f(x)$. The increment of the function

$$\Delta y = f(x) - f(0).$$

Hence

$$f(x) = \Delta y + f(0).$$

Since it is given that $\Delta y = k \cdot \Delta x = kx$ and $f(0) = b$, we have $f(x) = kx + b$ as required.

3°. It follows from (I) that

$$\boxed{\frac{\Delta y}{\Delta x} = k} \qquad \text{(II)}$$

i.e., *the ratio of the increment of a linear function to the increment of the argument does not depend on the value of the argument; it is constant for a given function and equal to the slope of the straight line which represents the graph of the function.*

The aforesaid ratio is called the rate of change of the linear function.

A linear function varies uniformly and the rate of change is numerically equal to the slope.

4°. A practical example of a linear function is the distance s travelled by a body moving with a uniform velocity v,

$$s = vt + s_0,$$

or the length l of a spring subject to the action of a stretching force P (Hooke's law):

$$l = kP + l_0,$$

or the volume v of a gas that varies with the temperature T,

$$v = 1 + \beta T,$$

etc.

Sec. 74. Examples in Finding Rates of Change

1°. Let the distance s travelled by a body freely falling in a vacuum during time t be given by the formula

$$s = \frac{1}{2} g t^2,$$

where g — the acceleration of gravity — is approximately equal to 9.8 m/sec^2.

Let us calculate the rate of free fall of such a body at $t = 2$ sec. To do this, let us determine the average velocity of the body over some interval of the time immediately succeeding $t = 2$ sec, for example, over an interval of 0.1 sec between $t = 2$ and $t_1 = 2.1$ sec ($t_1 - t = 0.1$ sec).

Let the distance travelled during t be s, and that during t_1, s_1. Then the distance travelled during the interval t_1-t is

$$s_1-s=\frac{1}{2}gt_1^2-\frac{1}{2}gt^2=\frac{1}{2}g(t_1+t)(t_1-t).$$

The average rate of fall during this interval is

$$v_{av}=\frac{s_1-s}{t_1-t}=\frac{1}{2}g(t_1+t).$$

Numerically,

$$v_{av}=\frac{1}{2}g(2.1+2)=2.05\,g.$$

Let us reduce the interval of time, t_1-t, step by step, each time by a factor of 10. We get the following table.

t	t_1	t_1+t	$v_{av}=\frac{1}{2}g(t_1+t)$	$v_{av}=A+\alpha$
2	2.1	4.1	$2.05g$	$2g+\frac{g}{20}$
2	2.01	4.01	$2.005g$	$2g+\frac{g}{200}$
2	2.001	4.001	$2.0005g$	$2g+\frac{g}{2000}$
2	2.0001	4.0001	$2.00005g$	$2g+\frac{g}{20000}$

In the last column, the average velocity is written as the sum of two terms: a constant $A=2g$ and a variable α equal to a fraction in which the numerator g is a definite number while the denominator increases without bound. Such a fraction is an infinitesimal (Sec. 47).

Thus, the variable average velocity is the sum of a constant $2g$ and an infinitesimal α. Therefore (Sec. 49) when $t_1-t\to 0$, the velocity has a limit equal to $2g$:

$$\lim_{t_1-t\to 0}\frac{s_1-s}{t_1-t}=2g.$$

The values of the average velocity, $2.05g$, $2.005g$, etc., calculated for the time intervals 0.1 sec, 0.01 sec, etc., are approximations to the value of the velocity at time $t=2$ sec, and this approximation is the closer to the actual velocity the smaller the interval of time taken. *The limit of the average velocity when t_1-t tends to zero is the velocity v at time t.*

Hence, at time $t=2$ sec, the velocity $v=2g\cdot$m/sec.

Note that we took times t_1 later than t; but we could just as well have taken t_1 prior to t: $t_1=1.9$; 1.99; 1.999, etc. Then the average velocity would be $1.95g$, $1.995g$, $1{,}9995g$, etc., with the same limit $2g$ (Sec. 44, 2°).

2°. The difference s_1-s is the increment of the distance and may be denoted by Δs (Sec. 65), while the difference t_1-t is the increment in time and may be denoted by Δt. We then have

$$1)\quad v_{av}=\frac{s_1-s}{t_1-t}=\frac{\Delta s}{\Delta t},$$

i.e., *the average velocity of a freely falling body is given by the ratio of the increment of the path to the increment of the time during which the increment in path takes place*;

$$2)\ v=\lim_{t_1-t\to 0}\frac{s_1-s}{t_1-t}=\lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t},$$

i.e., *the exact value of the velocity of a freely falling body at a particular instant is the limit of the ratio of increment of path to the increment of time when the latter increment tends to zero.*

3°. If a kilogram body absorbs one and the same quantity of heat (Q_1-Q cal) during a rise of different temperatures t by one and the same amount $(t_1-t)°$, then the relation

$$\frac{Q_1-Q}{t_1-t}=\frac{\Delta Q}{\Delta t}$$

has a constant value, and the amount of heat Q is a linear function of the temperature t. The ratio $\frac{\Delta Q}{\Delta t}$ in this case represents the rate of change in the quantity of heat Q for an arbitrary process of variation of temperature t and is called the *heat capacity* of the body.

Careful experiments have revealed that the quantity of heat Q absorbed by a substance is not a linear function of t. For example, the heat Q absorbed by 1 kg of iron in a temperature rise from 0° to $t=200°$ is given with sufficient accuracy by the formula

$$Q=0.1053\,t+0.000071\,t^2.$$

Let us use this formula to find the heat capacity of iron at $t=50°$.

Solution. To simplify the writing we shall denote 0.000071 by a and 0.1053 by b. Then

$$Q=bt+at^2.$$

Suppose that Q calories of heat are absorbed by 1 kg of iron when the latter is heated from 0° to $t°$, and $Q+\Delta Q$ calories are absorbed by it on being heated from 0° to $(t+\Delta t)°$:

$$Q=bt+at^2, \tag{1}$$

$$Q+\Delta Q=b\,(t+\Delta t)+a\,(t+\Delta t)^2. \tag{2}$$

Subtracting (1) from (2) we find the quantity (ΔQ) by which the quantity of heat Q changes when the temperature changes by $\Delta t°$:

$$\Delta Q = b \cdot \Delta t + 2at \cdot \Delta t + a \cdot \Delta t^2. \tag{3}$$

Dividing (3) by Δt we get the average rate of change of the quantity of heat Q (average heat capacity) over the interval $[t, t+\Delta t]$:

$$\frac{\Delta Q}{\Delta t} = b + 2at + a \cdot \Delta t. \tag{4}$$

We note that $b + 2at$ is constant for the given values of $b = 0.1053$, $a = 0.000071$, $t = 50°$:

$$b + 2at = C,$$

where $C = 0.1053 + 2 \cdot 0.000071 \cdot 50 = 0.1124$ kcal.

The term $a \cdot \Delta t$ of the sum is a variable quantity depending on the value of Δt. As Δt tends to zero, $a \cdot \Delta t$ becomes an infinitesimal, and the constant C becomes the limit of the average rate of change $\frac{\Delta Q}{\Delta t}$:

$$\lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = C.$$

This constant C represents the rate of change of the quantity of heat Q at a given temperature $t°$ for any arbitrary process of temperature variation and is called the heat capacity of the body at this temperature.

Thus, the heat capacity of iron at $t = 50°$ is $C = 0.1124$ kcal.

Deduction. *The rate of change of the quantity of heat absorbed by a body at a given temperature (the heat capacity of the body at that temperature) is the limit of the ratio of the increment of the quantity of heat to the increment of temperature when the latter tends to zero.*

Sec. 75. Derivative Function

Let us try to determine the rate of change of the quantity y as a function of the variation of x. Since we are interested in all possible cases, we shall regard the relation $y = f(x)$ and the quantities x and y as purely mathematical entities without assigning a physical meaning to the symbols. To find the rate of change of y with respect to x we shall use the same methods of calculation as were used in the previous section.

We consider a function $y = f(x)$ continuous over the interval $[a, b]$. Let us take two numbers in this interval: x and $x + \Delta x$. During the course of our discussion we shall treat x as some fixed quantity and Δx as its increment. Now, the increment Δx of the argument will cause an increment Δy in the function.

Thus,

$$\Delta y = f(x+\Delta x) - f(x). \tag{I}$$

The ratio of the increment Δy of the function to the increment Δx of the argument is found to be

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}. \tag{II}$$

From the foregoing, this ratio is the average rate of change in y with respect to x over the interval $[x, x+\Delta x]$.

Let Δx tend to zero without bound.

For a continuous function $f(x)$ the fact that Δx tends to zero causes Δy to tend to zero (Sec. 70), and the ratio (II) becomes a ratio of infinitesimals, generally speaking, a variable quantity. Let this variable ratio (II) have a definite limit * denoted by $f'(x)$.

$$\boxed{\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x)} \tag{III}$$

From the physical point of view, this limit is the value of the rate of change of the function $f(x)$ with respect to its argument for a given value x of this argument. In analysis, this limit is called the *derivative of the given function at the point x.*

Definition. *The limit of the ratio of the increment of a given function at x to the increment of its argument at that point as the latter tends to zero is called the derivative of the function at that point.*

2°. Let a definite value of the rate of change of the function $f(x)$ correspond to every value of the argument x. Then the rate of change $f'(x)$ is a new function of the argument x called the *derivative function* of the given function $f(x)$.

For example, the derivative function of the quadratic function

$$Q = bt + at^2$$

is a linear function:

$$Q' = b + 2at. \qquad \text{(Sec. 74, 3°)}$$

3°. A derivative function is denoted thus:

1) a prime is used, or 2) the symbol $\frac{d}{dx}$ is written before the symbol of the given function.

If the given function be denoted by the letter y, its derivative may be designated as follows:

1) y', read as "derivative of the function y", or

2) $\frac{dy}{dx}$, read as "dy with respect to dx", or "dy over dx".

If the function is given as $f(x)$, the derivative may be written as:

1) $f'(x)$, read as "derivative of the function $f(x)$", or "f prime of x" or

* We cannot assert that the ratio $\frac{\Delta y}{\Delta x}$ always has a definite limit.

2) $\frac{df(x)}{dx}$, read as "df of x with respect to dx".

4°. *Finding the derivative of a given function is called differentiating that function.*

The general rule for differentiation (for finding the derivative) is:

1) find the increment Δy of the function, i.e., find the difference of values of the function for values of the argument $x+\Delta x$ and x;

2) find the ratio $\frac{\Delta y}{\Delta x}$ by dividing the above-obtained equation by Δx;

3) find the limit of the ratio $\frac{\Delta y}{\Delta x}$ when $\Delta x \to 0$.

Example. Find the derivative of the function $y=x^3+1$ at any point x.

Solution.

1) $\Delta y=(x+\Delta x)^3+1-(x^3+1)$.

Solving we get

$$\Delta y=3x^2\cdot\Delta x+3x\cdot\Delta x^2+\Delta x^3;$$

2) $\frac{\Delta y}{\Delta x}=3x^2+3x\cdot\Delta x+\Delta x^2$;

3) $\frac{dy}{dx}=\lim\limits_{\Delta x\to 0}(3x^2+3x\cdot\Delta x+\Delta x^2)=3x^2+3x\cdot 0+0=3x^2$.

5°. Note that the derivative of a linear function $y=kx+b$ is a constant quantity, k.

Indeed, for the linear function $y=kx+b$

$$\Delta y=k\cdot\Delta x;$$

$$\frac{\Delta y}{\Delta x}=k;$$

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}k=k.$$

6°. Derivatives are often encountered in science and engineering. Some examples of derivatives:

1) in the motion of a body, the path traversed s is a function of the time t; the velocity of motion at time t is the derivative of the path s with respect to the time t, i.e.,

$$v=\frac{ds}{dt};$$

2) in the rotational motion of a rigid body (e.g., a flywheel) about the x-axis (Fig. 91) the angle of rotation φ is a function of the time t:

$$\varphi = f(t);$$

the angular velocity ω at any given time t is a derivative of the angle of rotation with respect to the time, i.e.,

$$\omega = \frac{d\varphi}{dt};$$

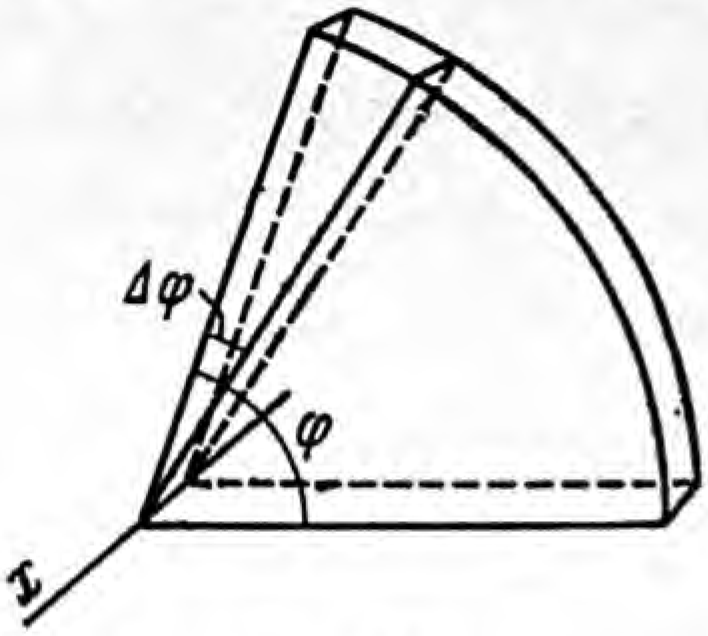

Fig. 91.

3) during cooling the temperature T of a body is a function of the time t,

$$T = f(t);$$

and the rate of cooling at any particular time t is a derivative of the temperature T with respect to the time t, i.e., $\frac{dT}{dt}$;

4) the heat capacity C of a body at a given temperature t is a derivative of the quantity of heat Q with respect to the temperature t,

$$C = \frac{dQ}{dt};$$

5) careful experiments show that when a rod is heated the increment in length, Δl, can only approximately be said to be proportional to the change in temperature Δt. For this reason, the function $l = f(t)$ is not a linear function, and the ratio $\frac{\Delta l}{\Delta t}$ is only the *average* coefficient of linear expansion over the interval $[t, t+\Delta t]$. The coefficient of linear expansion α, at a given temperature t, is a derivative of the length l with respect to the temperature t,

$$\alpha = \frac{dl}{dt}.$$

Sec. 76. Tangent to a Curve

1°. Take a point C on the straight line AB (Fig. 92) and draw through it a straight line CM not coincident with AB. Imagine that CM rotates about C in such manner that the angle γ between CM and AB tends to zero. Then the fixed line AB is called the *limiting position* of the moving line CM.

2°. Imagine that on the curve AB (Fig. 93) a point M approaches without bound a fixed point C on the curve, and the secant CM rotates about C. It may happen that irrespective of whether

the point M approaches C in the direction from A to C or in the direction from B to C (point M' in Fig. 93), there exists one and the same straight line CT, which is the limiting position of CM.

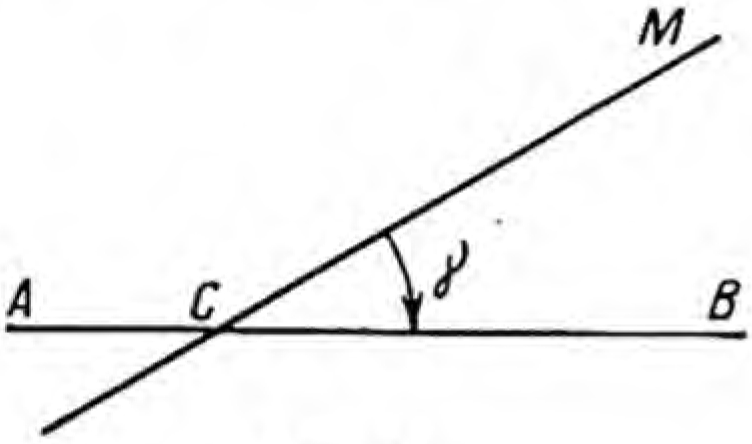

Fig. 92.

Definition. *The straight line CT, the limiting position of the secant CM, is called the tangent to the curve at the point C.*

Point C is called the *point of contact or tangency.*

3°. **Corollary.** *Angle φ (Fig. 93) formed by the tangent CT with the axis Ox is the limit of the angle α formed with the same axis*

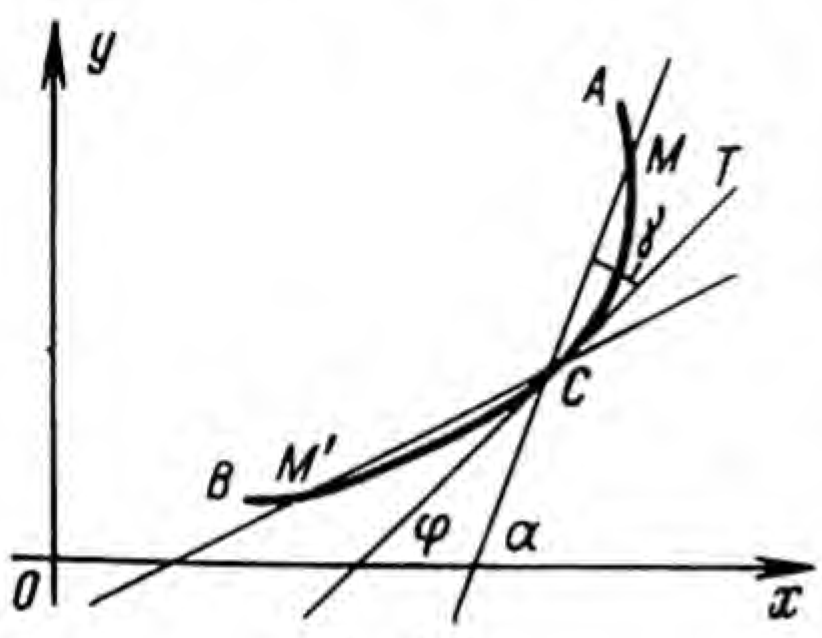

Fig. 93.

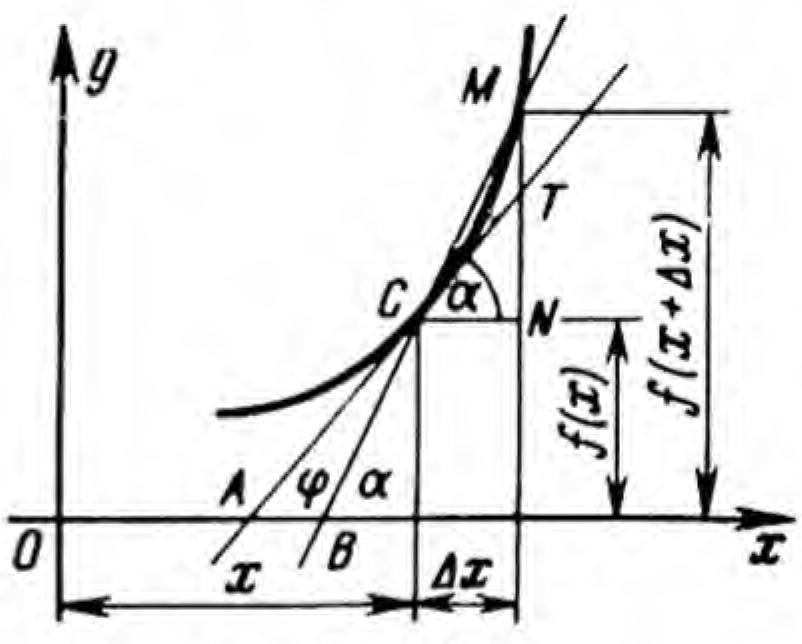

Fig. 94.

by the secant CM, for which the given tangent CT is the limiting position.

Indeed, angle γ between the tangent CT and the secant CM is equal to $\alpha - \varphi$ (Sec. 16):

$$\alpha - \varphi = \gamma.$$

According to the definition of a tangent, angle γ is an infinitesimal, and therefore

$$\varphi = \lim \alpha. \tag{I}$$

4°. **Theorem.** *If the line $y = f(x)$ has a tangent at the point x, not parallel to the axis Oy, then the slope of the tangent is equal to the value of the derivative $f'(x)$ at x.*

Proof. The slope of the tangent

$$\tan\varphi = \tan(\lim\alpha)$$

since, from the foregoing, $\varphi = \lim\alpha$.

Excluding the case of $\varphi = \frac{\pi}{2}$, we have, by virtue of the continuity of a tangent (Sec. 72),

$$\tan(\lim\alpha) = \lim\tan\alpha.$$

Hence, $\tan\varphi = \lim\tan\alpha$.
By formula (VI), Sec. 5, for CM (Fig. 94) we have

$$\tan\alpha = \frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

Passing to the limit when $\Delta x \to 0$ (point M approaches point C without bound at the angle $\alpha \to \varphi$ when $\Delta x \to 0$) we have

$$\lim_{\Delta x\to 0} \tan\alpha = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x).$$

Consequently,

$$\boxed{\tan\varphi = f'(x)} \tag{IV}$$

Sec. 77. Geometrical Meaning of a Derivative

1°. The converse theorem embodying the geometrical interpretation of a derivative is also true: *if a function $y=f(x)$ has a definite derivative at the point x, then:*

1) *a tangent to the graph of the function exists at this point,*
2) *the slope of the tangent is equal to the value of the derivative $f'(x)$ at x.*

Proof. It is assumed that the limit of the ratio $\frac{\Delta y}{\Delta x}$ exists. But the ratio $\frac{\Delta y}{\Delta x}$ is the tangent of the angle formed by the secant CM (Fig. 94).

$$\frac{\Delta y}{\Delta x} = \tan\alpha. \tag{1}$$

Hence, as is given, there exists $\lim\limits_{\Delta x\to 0}\tan\alpha = \tan\left(\lim\limits_{\Delta x\to 0}\alpha\right)$.

It follows from equation (1) that

$$\alpha = \arctan\frac{\Delta y}{\Delta x}.$$

As a consequence of the continuity of the arc tangent, we have

$$\lim_{\Delta x\to 0}\alpha = \lim_{\Delta x\to 0}\arctan\frac{\Delta y}{\Delta x} = \arctan\left(\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\right).$$

But it is given that $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ exists and is equal to $f'(x)$ Therefore

$$\lim_{\Delta x \to 0} \alpha = \operatorname{arc\,tan} f'(x).$$

Assuming $\operatorname{arc\,tan} f'(x) = \varphi$, we get

$$\lim_{\Delta x \to 0} \alpha = \varphi.$$

Thus, α has a limit, which means that there exists a straight line passing through point C that makes an angle with Ox equal to $\lim\limits_{\Delta x \to 0} \alpha$. Such a straight line is the tangent to the curve at the given point $C[x, f(x)]$ and its slope $\tan \varphi = f'(x)$.

2°. **Remarks.** 1. The slope k of the straight line $y = kx + b$ is called the slope of the line to the axis Ox. The slope of the tangent to the curve $y = f(x)$ at the point (x_1, y_1) is called the slope of the curve; it is equal to the derivative at this point, i.e., $\tan \varphi = f'(x_1)$.

2. If the tangent to the curve $y = f(x)$ at the point (x_1, y_1) forms with the axis Ox: a) an acute angle φ, then the derivative $f'(x_1) > 0$, since $\tan \varphi > 0$ (Fig. 95); b) an obtuse angle φ, then the derivative $f'(x_1) < 0$, since $\tan \varphi < 0$ (Fig. 96). If the tangent is parallel to the x-axis (Fig. 97), then angle $\varphi = 0$, $\tan \varphi = 0$ and $f'(x_1) = 0$.

When the tangent is perpendicular to the x-axis, the approach of α to $\frac{\pi}{2}$ can yield one and the same infinite limit both on the right and left: $\tan \varphi = +\infty$ (Fig. 98) or $\tan \varphi = -\infty$ (Fig. 99), or infinite limits of unlike sign (in Fig. 100 at C, the limit on the left is $\tan \varphi = +\infty$, while the limit on the right is $\tan \varphi = -\infty$). In the first case (points A and B in Figs. 98, 99), the finction $f(x)$ is said to have an infinite derivative at these points; in the second case (at C in Fig. 100), there is no derivative, either finite or infinite.

Note that infinite derivatives are considered only at points of continuity of the function $f(x)$.

3. If the function $f(x)$ has a finite derivative at a point x, the function is said to be *differentiable* at this point. The function $f(x)$ is differentiable over the interval $a < x < b$, if its derivative $f'(x)$ is finite at every point of this interval.

4. A curve having a tangent is sometimes situated on both sides of the tangent (Figs. 97, B; 98, 99). The tangent in such a case is said to cross the curve.

4°. The straight line perpendicular to the tangent and passing through the point of contact is called the *normal* to the curve. According to the condition for mutual perpendicularity of two straight lines, the slope of the normal is $-\frac{1}{f'(x_1)}$.

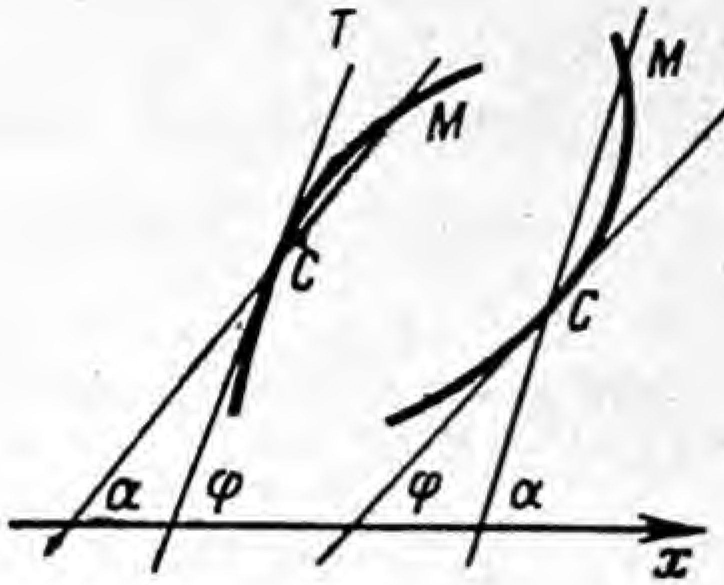

Fig. 95.

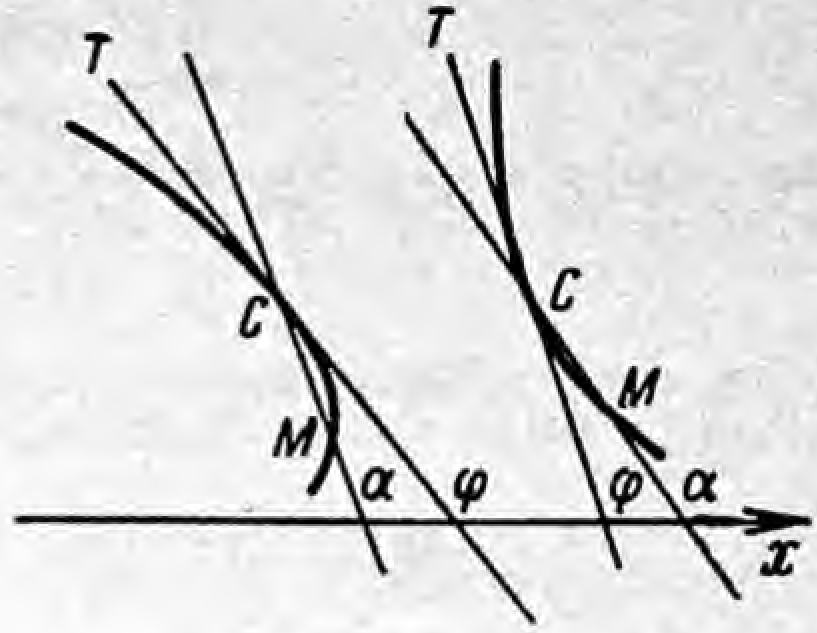

Fig. 96.

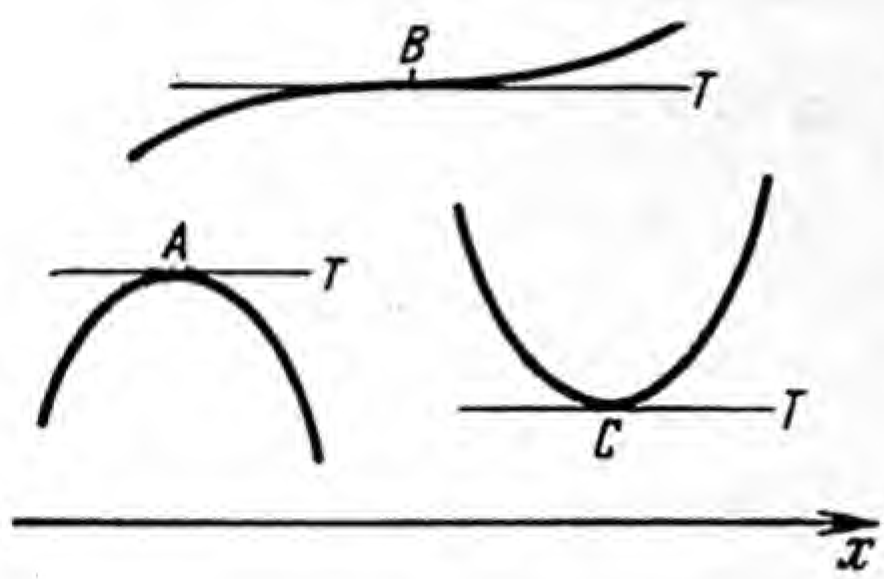

Fig. 97.

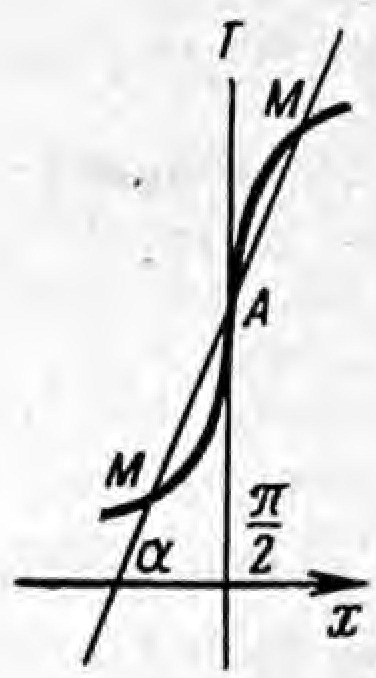

Fig. 98.

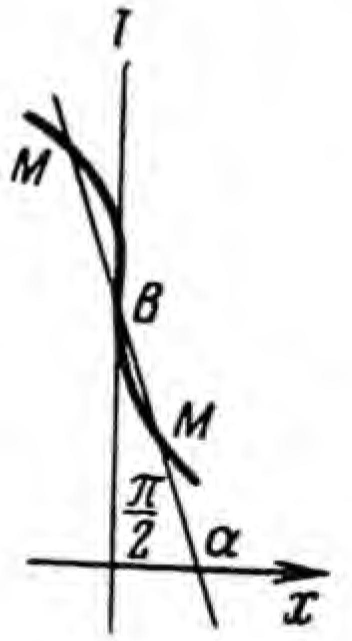

Fig. 99.

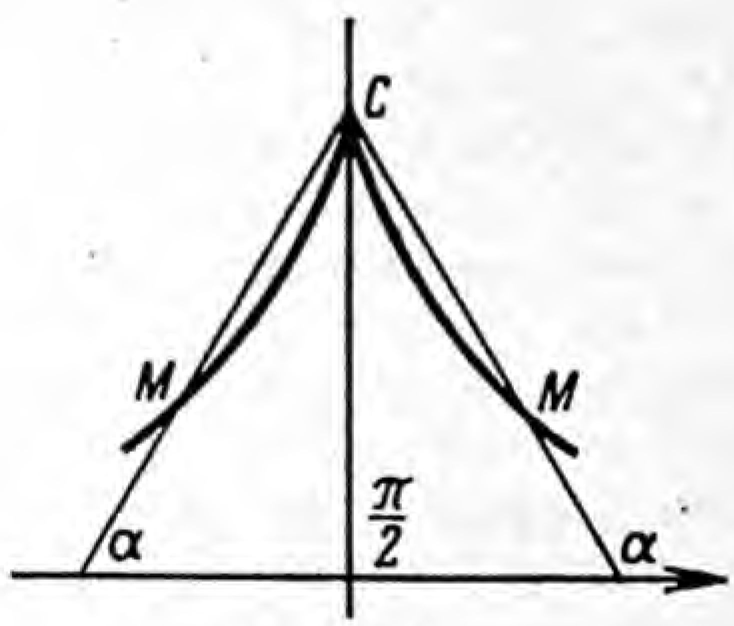

Fig. 100.

Sec. 78. Relationship Between Differentiability and Continuity of a Function

1°. **Theorem.** *If the function $y = f(x)$ has a definite derivative at the point x, then the function is continuous at this point.*

Proof. We write the identity

$$\Delta y = \frac{\Delta y}{\Delta x} \cdot \Delta x,$$

since it is always understood that $\Delta x \neq 0$. When Δx tends to zero, the ratio $\frac{\Delta y}{\Delta x}$ has a definite limit (given) and, consequently (Sec. 46), is a finite quantity. Δx is an infinitesimal. Hence, the product $\frac{\Delta y}{\Delta x} \cdot \Delta x$ is an infinitesimal with limit equal to zero:

$$\lim_{\Delta x \to 0} \Delta y = 0.$$

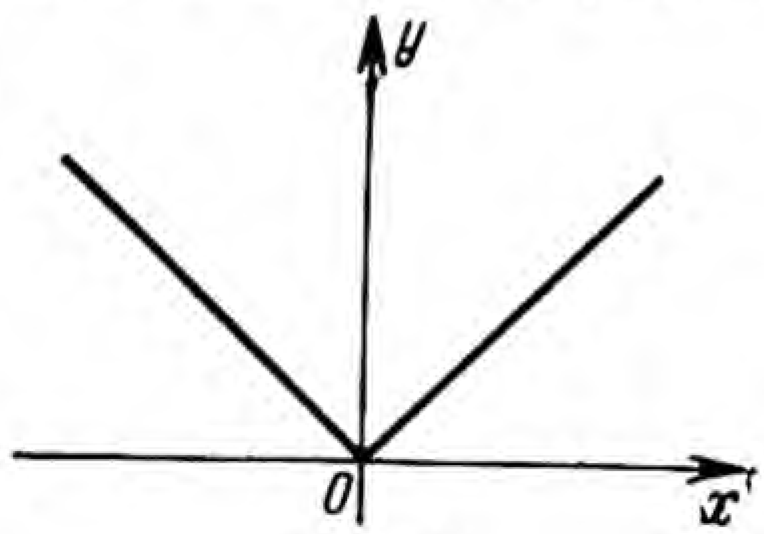

Fig. 101.

Consequently, the given function $y = f(x)$ is continuous (Sec. 70).

2°. The converse theorem is not true: *a continuous function may not have a derivative.* For example, the function

$$y = |x|$$

(Fig. 101) at the point $x = 0$ is continuous. However, no definite tangent exists at $x = 0$: the function is not differentiable.

3°. **Corollary.** *A function has no derivative at a point of discontinuity.*

The first clear-cut distinction between the concept of continuity and differentiability was given by the brilliant Russian scientist N. I. Lobachevsky.

CHAPTER VII

DERIVATIVES OF ELEMENTARY FUNCTIONS

Sec. 79. Preliminary Remarks

1°. In a simultaneous examination of two or more functions it will be assumed that each one of them is differentiable, i.e., that it has a definite derivative at any arbitrary point x.

2°. The formulas for finding derivatives are also called formulas of differentiation.

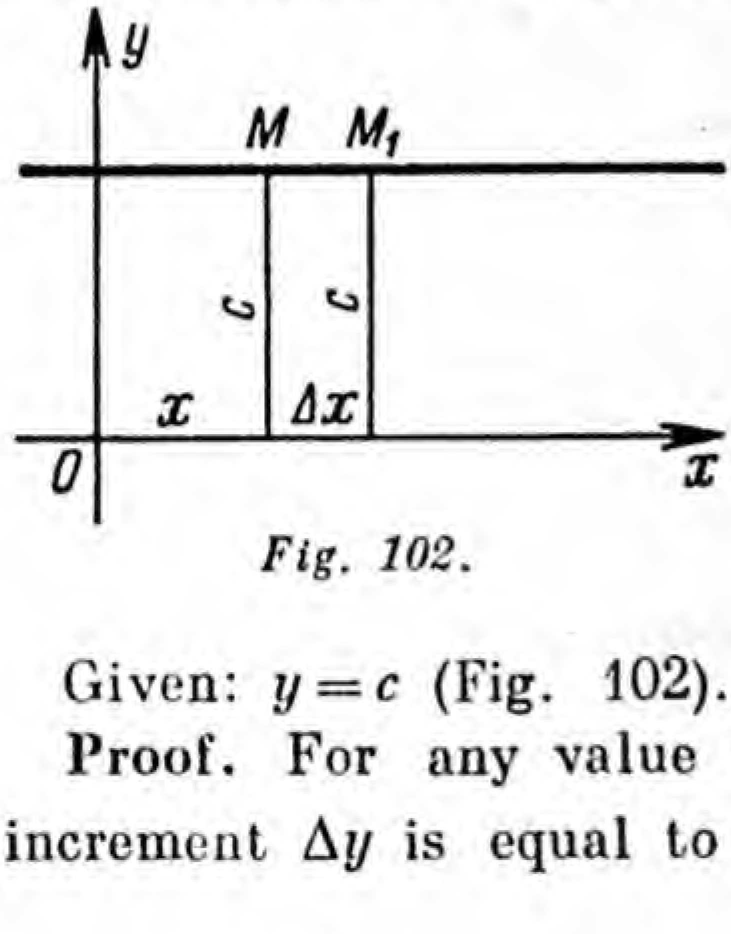

Fig. 102.

Sec. 80. The Derivative of a Constant

Theorem. *A constant function has a derivative equal to zero at any point x.*

Given: $y=c$ (Fig. 102). To prove: $c'=0$.

Proof. For any value of x and for any increment Δx, the increment Δy is equal to zero. The ratio $\frac{\Delta y}{\Delta x}=0$, i.e.,

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=0;$$

$$\boxed{c'=0} \qquad \text{(I)}$$

Sec. 81. The Derivative of a Power

1°. If a function is defined by the equation $y=x^n$, where the exponent n is any real number, the function is called a *power function.*

The exponent n may be positive or negative, rational (i.e., integer or fraction) or irrational.

2°. **Theorem.** *The derivative of the power of an independent variable is equal to the exponent multiplied by the independent*

variable to the same power minus one,

$$\text{i.e., if } y = x^n,\ y' = (x^n)' = n \cdot x^{n-1}.$$

Proof. Let n be a positive integer. The increment of the function

$$\Delta y = (x + \Delta x)^n - x^n.$$

Using Newton's binomial formula, we get

$$\Delta y = x^n + \frac{n}{1} x^{n-1} \Delta x + \frac{n(n-1)}{1 \cdot 2} x^{n-2} \cdot \Delta x^2 + \\ + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^{n-3} \Delta x^3 + \ldots + \Delta x^n - x^n,$$

or

$$\Delta y = n \cdot x^{n-1} \Delta x + \frac{n(n-1)}{1 \cdot 2} x^{n-2} \Delta x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^{n-3} \Delta x^3 + \\ + \ldots + \Delta x^n.$$

Dividing through by Δx, we get the relation

$$\frac{\Delta y}{\Delta x} = n \cdot x^{n-1} + \frac{n(n-1)}{1 \cdot 2} x^{n-2} \Delta x + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^{n-3} \cdot \Delta x^2 + \\ + \ldots + \Delta x^{n-1}.$$

And passing to the limit at $\Delta x \to 0$:

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = n \cdot x^{n-1}.$$

Consequently,

$$\boxed{(x^n)' = n \cdot x^{n-1}} \qquad \text{(II)}$$

Remark. The theorem is true also for negative, fractional, and irrational exponents. Proof will be given below. Let us start using this formula for all constant exponents.

3° **Examples.** 1. Find the derivative of the function $y = x^4$.

5 **Solution.** By formula (II) $y' = 4 \cdot x^3$.

2. Find the derivative of the function $y = \sqrt[3]{x^2}$.

Solution. Writing the radical as a fractional power, we get $y = x^{\frac{2}{3}}$. By formula (II),

$$y' = \frac{2}{3} \cdot x^{\frac{2}{3} - 1} = \frac{2}{3} x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}.$$

3. Find the derivative of the function $y = \frac{1}{x^3}$.

Solution. Writing the fraction as a negative power, we get $y = x^{-3}$. By formula (II),

$$y' = -3 \cdot x^{-3-1} = -3 \cdot x^{-4} = -\frac{3}{x^4}.$$

4. Find the derivative of the function $y=\frac{1}{\sqrt{x}}$.

Solution. $y=x^{-\frac{1}{2}}$.
By formula (II),

$$y'=-\frac{1}{2}x^{-\frac{3}{2}}=-\frac{1}{2x\sqrt{x}}.$$

4°. **Corollary.** *The derivative of an independent variable is equal to unity* since

$$x'=(x^1)'=1\cdot x^{1-1}=x^0=1.$$

Sec. 82. The Derivative of the Product of a Constant and a Function

1°. **Theorem.** *The derivative of the product of a constant with a function is equal to the product of the constant into the derivative of the function,* i.e., if $y=c\cdot f(x)$, where c is a constant, then $y'=c\cdot f'(x)$.

Proof. First we find Δy, i.e., the difference between the values of the given function $c\cdot f(x)$ for values of the argument $x+\Delta x$ and x:

$$\Delta y=c\cdot f(x+\Delta x)-c\cdot f(x),$$

or

$$\Delta y=c\,[f(x+\Delta x)-f(x)].$$

We find the ratio $\frac{\Delta y}{\Delta x}$ by dividing through the above equation by Δx:

$$\frac{\Delta y}{\Delta x}=c\cdot\frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

We then find the derivative (applying Sec. 52, 2°):

$$y'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=c\cdot\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=c\cdot f'(x).$$

Therefore,

$$\boxed{[c\cdot f(x)]'=c\cdot f'(x)} \qquad \textbf{(III)}$$

which is what is required.

The proved theorem is sometimes stated as follows: *in differentiation, the numerical factor may be taken out of the differentiation sign.*

2°. **Examples.** 1. Find the derivative of the function $y=2\cdot x^4$.
Solution. By formula (III), $y'=2\cdot(x^4)'$ and by formula (II),

$$y'=2\cdot 4x^3=8x^3.$$

2. Find the derivative of the function $y=\frac{4}{\sqrt{x}}$.

Solution. Converting the fraction and the radical into a negative fractional power, we get

$$y=4\cdot x^{-\frac{1}{2}}.$$

By formula (III),

$$y'=4\cdot\left(x^{-\frac{1}{2}}\right)'=4\cdot\left(-\frac{1}{2}\right)\cdot x^{-\frac{3}{2}}=-\frac{2}{x\sqrt{x}}.$$

Sec. 83. The Derivative of an Algebraic Sum of Functions

1°. **Theorem.** *The derivative of an algebraic sum of functions is equal to the algebraic sum of the derivatives of those functions.*

Let us prove this for the sum of three functions: $u(x)+v(x)-z(x)$. For the sake of brevity we shall denote the functions simply as u, v and z. Thus: $y=u+v-z$.

Let us take any arbitrary value of x and change it by Δx. Then u, v, and z will each change their values by Δu, Δv and Δz, respectively; this will cause the function y to change its value by Δy.

$$y+\Delta y=u+\Delta u+v+\Delta v-(z+\Delta z),$$

$$\Delta y=u+\Delta u+v+\Delta v-(z+\Delta z)-(u+v-z).$$

Removing brackets, we get

$$\Delta y=\Delta u+\Delta v-\Delta z.$$

Dividing this equation through by Δx, we obtain the relation $\frac{\Delta y}{\Delta x}$:

$$\frac{\Delta y}{\Delta x}=\frac{\Delta u}{\Delta x}+\frac{\Delta v}{\Delta x}-\frac{\Delta z}{\Delta x}.$$

We find the derivative

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}+\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}-\lim_{\Delta x\to 0}\frac{\Delta z}{\Delta x},$$

or

$$\boxed{y'=[u(x)-v(x)-z(x)]'=u'(x)+v'(x)-z'(x)} \qquad \text{(IV)}$$

as required.

2°. **Example.** Find the derivative of the function

$$y=\frac{3}{x^2}-\sqrt{x}+7.$$

Solution. Introducing negative and fractional exponents, we get

$$y=3x^{-2}-x^{\frac{1}{2}}+7.$$

y is the algebraic sum of three functions: $3x^{-2}=u$, $x^{\frac{1}{2}}=v$ and $7=z$. By formula (IV), $y'=(3x^{-2})'-(x^{\frac{1}{2}})'+(7)'$. By formulas (III), (II), (I),

$$y'=-6x^{-3}-\frac{1}{2}x^{-\frac{1}{2}}+0 \quad \text{or} \quad y'=-\left(\frac{6}{x^3}+\frac{1}{2\sqrt{x}}\right).$$

Sec. 84. The Derivative of a Product of Functions

1°. **Theorem 1.** *The derivative of the product of two functions is equal to the derivative of the first function multiplied by the second function plus the derivative of the second function multiplied by the first function*, i.e., if $y=u(x)\cdot v(x)$, or, concisely, $y=u\cdot v$, then $y'=(u\cdot v)'=u'v+v'u$.

Proof. Taking some arbitrarily definite value of x of the argument, let us give it the increment Δx. Then the functions u, v and y will also receive corresponding increments Δu, Δv and Δy:

$$y+\Delta y=(u+\Delta u)\cdot(v+\Delta v),$$

and the increment in the function y will be

$$\Delta y=(u+\Delta u)(v+\Delta v)-u\cdot v.$$

Removing the brackets and collecting terms, we get

$$\Delta y=v\cdot\Delta u+u\cdot\Delta v+\Delta u\cdot\Delta v.$$

Dividing through by Δx, we obtain the ratio $\frac{\Delta y}{\Delta x}$:

$$\frac{\Delta y}{\Delta x}=v\cdot\frac{\Delta u}{\Delta x}+u\cdot\frac{\Delta v}{\Delta x}+\Delta u\cdot\frac{\Delta v}{\Delta x}.$$

Passing to the limit when $\Delta x\to 0$, we get

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}v\cdot\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}+\lim_{\Delta x\to 0}u\cdot\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}+\lim_{\Delta x\to 0}\Delta u\cdot\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}.$$

Remember that u and v signify the numerical values of the functions $u(x)$ and $v(x)$ for some definite value of x, i.e., u and v do not depend on Δx. Therefore $\lim\limits_{\Delta x\to 0}v=v$ and $\lim\limits_{\Delta x\to 0}u=u$.

Under the condition given in Sec. 79, the functions u and v possess derivatives and, hence, are continuous (Sec. 78). Since u is continuous (Sec. 70), $\lim\limits_{\Delta x\to 0}\Delta u=0$.

By definition:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = y', \quad \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = u' \quad \text{and} \quad \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = v'.$$

Consequently,

$$y' = v \cdot u' + u \cdot v' \quad \text{or} \quad \boxed{y' = (u \cdot v)' = u'v + v'u} \tag{V}$$

as required.

2°. **Example.** Find the derivative of the function

$$y = (2x^2 + 3x) \cdot (3 - 2x).$$

Solution. The given function is the product of two functions:

$$2x^2 + 3x = u \text{ and } 3 - 2x = v.$$

By formula (V) we find

$$y' = (2x^2 + 3x)' \cdot (3 - 2x) + (3 - 2x)' \cdot (2x^2 + 3x) =$$
$$= (4x + 3) \cdot (3 - 2x) + (-2) \cdot (2x^2 + 3x).$$

Removing the brackets and simplifyng, we get

$$y' = 3(3 - 4x^2).$$

3°. **Theorem 2.** *The derivative of the product of n functions is equal to the sum of n products obtained by multiplying the derivative of each of these functions by all the remaining functions unchanged, i.e.,*

$$\boxed{\begin{gathered}(u_1 \cdot u_2 \ldots u_n)' = u_1' u_2 \cdot \ldots u_n + \\ + u_2' u_1 u_3 \ldots u_n + \ldots + u_n' u_1 u_2 \ldots u_{n-1}\end{gathered}} \tag{VI}$$

Proof. Let us take the product of three functions $u_1 \cdot u_2 \cdot u_3$. Treating the product $u_1 \cdot u_2$ as one function and differentiating by formula (V), we find

$$(u_1 u_2 u_3)' = [(u_1 u_2) \cdot u_3]' = (u_1 u_2)' u_3 + u_3' (u_1 u_2) =$$
$$= (u_1' u_2 + u_2' u_1) \cdot u_3 + u_3' u_1 u_2 = u_1' u_2 u_3 + u_2' u_1 u_3 + u_3' u_1 u_2.$$

Similarly, treating the product of three functions, $u_1 u_2 u_3$, as one function and then differentiating, we find the derivative of the product of four functions, and so forth.

In general, it is possible to prove that if formula (VI) is true for a product of m factors, it is also true for a product of $m+1$ factors.

Consequently the theorem holds for any number of factors n.

Sec. 85. The Derivative of a Fraction

1°. **Theorem.** *The derivative of a fraction is equal to a fraction, the numerator of which is the derivative of the numerator multiplied by the denominator minus the derivative of the denominator multi-*

plied by the numerator, while the denominator is the square of the given denominator, i.e., if

$$y=\frac{u(x)}{v(x)} \quad \text{or,}$$

concisely, $y=\frac{u}{v}$, then $y'=\left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$.

Proof. Let us take an arbitrarily definite value of x and let the value of the denominator v be different from zero. Altering the value of x by the increment Δx, we get increments Δu, Δv and Δy of the functions u, v and y, respectively,

$$y+\Delta y=\frac{u+\Delta u}{v+\Delta v},$$

and the following increment of the function y:

$$\Delta y=\frac{u+\Delta u}{v+\Delta v}-\frac{u}{v}.$$

Reducing to a common denominator and doing the subtraction, we get

$$\Delta y=\frac{v\cdot\Delta u-u\cdot\Delta v}{v(v+\Delta v)}.$$

Dividing the numerator $v\Delta u-u\cdot\Delta v$ by Δx, we find the ratio $\frac{\Delta y}{\Delta x}$:

$$\frac{\Delta y}{\Delta x}=\frac{v\cdot\frac{\Delta u}{\Delta x}-u\cdot\frac{\Delta v}{\Delta x}}{v(v+\Delta v)}.$$

We then find the derivative:

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{\lim\limits_{\Delta x\to 0} v\cdot\lim\limits_{\Delta x\to 0}\frac{\Delta u}{\Delta x}-\lim\limits_{\Delta x\to 0} u\cdot\lim\limits_{\Delta x\to 0}\frac{\Delta v}{\Delta x}}{\lim\limits_{\Delta x\to 0} v\cdot(\lim\limits_{\Delta x\to 0} v+\lim\limits_{\Delta x\to 0}\Delta v)}.$$

As we have already observed in the preceding section,

$$\lim_{\Delta x\to 0} v=v;\quad \lim_{\Delta x\to 0} u=u;\quad \lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}=u';\quad \lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}=v';$$
$$\lim_{\Delta x\to 0}\Delta v=0.$$

Hence

$$y'=\frac{v\cdot u'-u\cdot v'}{v^2}$$

or

$$\boxed{y'=\left(\frac{u}{v}\right)=\frac{u'v-v'u}{v^2}} \qquad \textbf{(VII)}$$

as required.

2°. **Example.** Find the derivative of the function

$$y=\frac{x^2+7}{x^2-7}.$$

Solution. Putting $x^2+7=u$, $x^2-7=v$, we find by formula (VII)

$$y'=\frac{(x^2+7)'\cdot(x^2-7)-(x^2-7)'\cdot(x^2+7)}{(x^2-7)^2}$$

or

$$y'=\frac{2x(x^2-7)-2x(x^2+7)}{(x^2-7)^2}=\frac{2x(x^2-7-x^2-7)}{(x^2-7)^2}=-\frac{28x}{(x^2-7)^2}.$$

3°. **Corollary.** 1. If the denominator of a fraction is a constant c, then

$$y=\frac{u(x)}{c}=\frac{1}{c}\cdot u(x),$$

and the derivative is found as the derivative of an integral function. In this case,

$$\boxed{y'=\left[\frac{u(x)}{c}\right]'=\frac{u'(x)}{c}} \qquad \text{(VIII)}$$

i.e., *if the denominator of a fraction is a number c, the derivative of the fraction is equal to the derivative of the numerator divided by c.*

2. If the numerator of a fraction is a number c, i.e.,

$$y=\frac{c}{v(x)}=\frac{c}{v},$$

then by formula (VII)

$$y'=\left(\frac{c}{v}\right)'=\frac{c'v-v'c}{v^2}=\frac{0\cdot v-v'\cdot c}{v^2}=-\frac{c\cdot v'}{v^2}$$

or

$$\boxed{y'=\left[\frac{c}{v(x)}\right]'=-\frac{c\cdot v'(x)}{v^2(x)}} \qquad \text{(IX)}$$

i.e., *if the numerator of a fraction is a number c, the derivative is equal to minus that number multiplied by the derivative of the denominator and divided by the square of the denominator.*

4°. **Example.** Find the derivative of the function

$$y=\frac{3x^4}{4a}-\frac{3a^2}{x^2}.$$

Solution.

$$y'=\left(\frac{3x^4}{4a}\right)'-\left(\frac{3a^2}{x^2}\right)'.$$

The derivatives of fractions are found by formulas (VIII) and (IX) respectively.

Thus we get

$$y'=\frac{(3x^4)'}{4a}-\frac{-3a^2(x^2)'}{(x^2)^2}=\frac{3\cdot 4x^3}{4a}+\frac{3a^2\cdot 2x}{x^4}=\frac{3x^3}{a}+\frac{6a^2}{x^3}.$$

Sec. 86. Remarks

We have stated before that a function is said to be differentiable if it has a derivative. From theorems (formulas II to IX) it follows that a *function obtained by the addition, subtraction, multiplication or division of differentiable functions is itself a differentiable function.*

Sec. 87. The Function of a Function

1°. **Examples.** 1. Let $y=\sqrt{2px}$. If we put

$$2px=u,$$

it becomes evident that u is a function of x, for instance,

$$u=\varphi(x).$$

Then the given function

$$y=\sqrt{2px}=\sqrt{u}$$

is a function of u, which, in turn, is a function of x. Writing symbolically, if $y=f(u)$ and $u=\varphi(x)$, then

$$y=f[\varphi(x)].$$

2. $y=\sin x^3$ is a function of a function because if we put $x^3=u$, then $u=\varphi(x)$. And so $y=\sin u=\sin\varphi(x)$ is a function of a function.

3. More complex functions of functions are also met with in practice. For example: $y=\log(\sin x^3)$.

Here $\sin x^3$ is a function of a function, and this in turn is the argument of the log function. In this case, the logarithm is a function of $\sin u$, $\sin u$ is a function of u, and u itself is a function of x.

Symbolically, this function may be written

$$y=F\{f[\varphi(x)]\}.$$

The function of a function is called a *composite function.*

Sec. 88. The Derivative of a Function of a Function

1°. **Theorem.** *If y is a differentiable function of u and u is a differentiable function of x, then*

1) *y is a differentiable function with respect to x, and*

2) *the derivative of y with respect to x is equal to the product of the derivative of y with respect to u into the derivative of u with respect to x*, i.e., if

$$y = f(u), \quad \text{and} \quad u = \varphi(x), \quad \text{then} \quad \frac{dy}{dx} = f'(u) \cdot \varphi'(x).$$

Proof. Take a definite value x to which there corresponds a definite value u; in turn, to the value u there corresponds a definite value y. We give x an arbitrary increment Δx. This increment will cause an increment Δu in u, which in turn will cause an increment Δy in y. We shall assume that Δu and Δx always differ from zero*, i.e.,

$$\Delta u \neq 0 \quad \text{and} \quad \Delta x \neq 0.$$

Now, it is given, by the theorem, that $\frac{\Delta y}{\Delta u}$ and $\frac{\Delta u}{\Delta x}$ have limits equal, respectively, to the derivatives $f'(u)$ and $\varphi'(x)$, i.e.,

$$\lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} = f'(u) \quad \text{and} \quad \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \varphi'(x). \tag{1}$$

The functions $f(u)$ and $\varphi(x)$ are differentiable, which means (Sec. 78) that they are continuous, and hence the fact that Δx tends to zero will cause Δu and Δy to tend to zero (Sec. 70). Consequently, in equation (1), $\Delta u \to 0$ may be replaced by $\Delta x \to 0$. Then equalities (1) may be written as

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} = f'(u) \quad \text{and} \quad \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \varphi'(x). \tag{2}$$

Multiplying the two equalities together, we get

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = f'(u) \cdot \varphi'(x). \tag{3}$$

Applying the theorem on the limit of a product (Sec. 52), we get

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \right) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

Putting $\frac{dy}{dx}$ in place of the left-hand side of equation (3), we find that

$$\frac{dy}{dx} = f'(u) \cdot \varphi'(x),$$

as required.

---

* The theorem can be shown to be true even in those cases where Δu becomes zero for certain values of Δx, however small the latter may be.

The following alternative form of this formula is convenient for practical use:

$$\boxed{\frac{dy}{dx}=f'_u(u)\cdot u'_x} \tag{X}$$

This form is obtained by replacing $\varphi'(x)$ by u'_x.

2°. In the same manner we can show that formula (X) is applicable even when the given function is made up of a larger number of intermediate functions. Thus, for example, it can be shown that if $y=f(u)$, where $u=\varphi(t)$, and $t=\psi(x)$, then

$$\frac{dy}{dx}=f'(u)\cdot\varphi'(t)\cdot\psi'(x).$$

3°. **Example.** Find the derivative of the function $y=(ax+b)^n$.

Solution. Putting $ax+b=u$, we see that $y=u^n$ is a function of a function. Therefore, differentiating with respect to the variable u as a power function, and at the same time applying formula (X), we get

$$y'=n\cdot u^{n-1}\cdot u'.$$

Replacing u by $ax+b$, we have

$$y'=n\cdot(ax+b)^{n-1}\cdot(ax+b)'$$

or, since $(ax+b)'=a$,

$$y'=an(ax+b)^{n-1}.$$

4°. $$\boxed{\text{If } y=u^n,\text{ where } u=\varphi(x),\text{ then } y'=n\cdot u^{n-1}\cdot u'} \tag{XI}$$

i.e., *the derivative of the power of the argument in a composite function is equal to the product of its exponent into the base to the power less by unity and into the derivative of the base.*

Formula (X) is basic to the technique of differentiation.

5°. **Example.** Find the derivative of the function

$$y=(5+3x^3)^{10}\cdot(3x^2+8)^5.$$

Solution. Differentiating with the aid of formula (V) for the derivative of a product, we have

$$y'=[(5+3x^3)^{10}]'\cdot(3x^2+8)^5+[(3x^2+8)^5]'\cdot(5+3x^3)^{10}.$$

The derivatives $[(5+3x^3)^{10}]'$ and $[(3x^2+8)^5]'$ can be found from formula (XI):

$$y'=10(5+3x^3)^9\cdot 9x^2\cdot(3x^2+8)^5+5(3x^2+8)^4\cdot 6x\cdot(5+3x^3)^{10}.$$

Taking common factors out of the brackets, we have

$$y'=30x(5+3x^3)^9\cdot(3x^2+8)^4\cdot[3x(3x^2+8)+(5+3x^3)].$$

Performing the operations in the square brackets, we get

$$y'=30x(5+3x^3)^9\cdot(3x^2+8)^4\cdot(12x^3+24x+5).$$

6°. **Example.** Find the derivative of the function

$$y=\frac{2x-1}{\sqrt{3-x^2}}.$$

Solution. Applying formula (VII) for the differentiation of a fraction, we have

$$y=\frac{2x-1}{(3-x^2)^{\frac{1}{2}}}$$

$$y'=\frac{(2x-1)'\cdot(3-x^2)^{\frac{1}{2}}-[(3-x^2)^{\frac{1}{2}}]'\cdot(2x-1)}{[(3-x^2)^{\frac{1}{2}}]^2}.$$

The derivative of $(3-x^2)^{\frac{1}{2}}$ is found from formula (XI):

$$y'=\frac{2(3-x^2)^{\frac{1}{2}}-\frac{1}{2}(3-x^2)^{-\frac{1}{2}}\cdot(-2x)(2x-1)}{3-x^2}=$$

$$=\frac{2\cdot\sqrt{3-x^2}+\frac{x(2x-1)}{\sqrt{3-x^2}}}{3-x^2}=\frac{\frac{2(3-x^2)+x(2x-1)}{\sqrt{3-x^2}}}{3-x^2}=$$

$$=\frac{6-2x^2+2x^2-x}{(3-x^2)\sqrt{3-x^2}}=\frac{6-x}{(3-x^2)\sqrt{3-x^2}}.$$

Sec. 89. The Limit of the Ratio of a Sine to an Arc

Theorem. *The limit of the ratio of sin α to arc α when the arc α tends to zero is 1*, i.e.,

$$\lim_{\alpha\to 0}\frac{\sin\alpha}{\alpha}=1.$$

Proof. Take an arc BE (Fig. 103) less than a semicircle and draw the chord BE, a radius OA perpendicular to the chord BE, and the tangents BD and ED to the circle at points B and E. From the equality of the right-angled triangles OBD and OED it follows that the tangents have a common point D and that $BD=DE$.

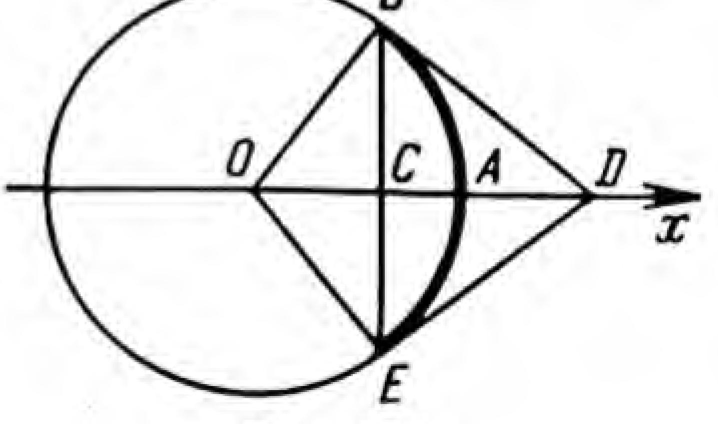

Fig. 103.

It is known from elementary geometry that the length of the arc BE of a circle: 1) is greater than its chord BE, but 2) is smaller than the perimeter of the broken line BDE described about this arc and having common ends with it. Thus,

$$2BC<2\smile AB<2BD.$$

Whence,

$$BC < \smile AB < BD.$$

Dividing the inequalities by the length of the radius R of the given arc, we have

$$\frac{BC}{R} < \frac{AB}{R} < \frac{BD}{R}.$$

The ratio $\frac{\smile AB}{R}$ is the radian measure of the arc AB; let us denote it by α. Then the ratio $\frac{BC}{R} = \sin\alpha$ and $\frac{BD}{R} = \tan\alpha$. Therefore,

$$\sin\alpha < \alpha < \tan\alpha.$$

Dividing the inequalities by $\sin\alpha$, we get

$$1 < \frac{\alpha}{\sin\alpha} < \frac{1}{\cos\alpha},$$

or

$$1 > \frac{\sin\alpha}{\alpha} > \cos\alpha.$$

When $\alpha \to 0$, the limit of 1 and the limit of $\cos\alpha = 1$, and since $\frac{\sin\alpha}{\alpha}$ is between 1 and $\cos\alpha$, hence (Sec. 56) its limit must also be equal to 1.

Consequently,

$$\lim_{\alpha\to 0} \frac{\sin\alpha}{\alpha} = 1.$$

Sec. 90. Derivatives of Trigonometric Functions

1°. *The function sin x has, at any point x, a derivative equal to cos x*, i.e., $(\sin x)' = \cos x$.

Proof. If $y = \sin x$, then $\Delta y = \sin(x + \Delta x) - \sin x$. Using the formula $\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}$, we get

$$\Delta y = 2\cos\left(x + \frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}.$$

Let us now find the ratio $\frac{\Delta y}{\Delta x}$ $\left(\text{for this it is sufficient to divide one factor, } \sin\frac{\Delta x}{2}, \text{ by } \Delta x\right)$:

$$\frac{\Delta x}{\Delta y} = 2\cos\left(x + \frac{\Delta x}{2}\right)\cdot\frac{\sin\frac{\Delta x}{2}}{\Delta x},$$

The derivative

$$\frac{dy}{dx}=2\cdot\lim_{\Delta x\to 0}\cos\left(x+\frac{\Delta x}{2}\right)\cdot\lim_{\Delta x\to 0}\frac{\sin\frac{\Delta x}{2}}{\Delta x}.$$

Since the cosine is a continuous function, by Sec. 72, 1°, we have

$$\lim_{\Delta x\to 0}\cos\left(x+\frac{\Delta x}{2}\right)=\cos\lim_{\Delta x\to 0}\left(x+\frac{\Delta x}{2}\right)=\cos x.$$

In order to find $\lim\limits_{\Delta x\to 0}\dfrac{\sin\frac{\Delta x}{2}}{\Delta x}$, we put $\dfrac{\Delta x}{2}=\alpha$.

Then $\Delta x=2\alpha$ and $\alpha\to 0$ if $\Delta x\to 0$:

$$\lim_{\Delta x\to 0}\frac{\sin\frac{\Delta x}{2}}{\Delta x}=\lim_{\alpha\to 0}\frac{\sin\alpha}{2\alpha}=\frac{1}{2}\cdot\lim_{\alpha\to 0}\frac{\sin\alpha}{\alpha}=\frac{1}{2}\cdot 1=\frac{1}{2}.$$

Consequently, $\dfrac{dy}{dx}=2\cdot\cos x\cdot\dfrac{1}{2}=\cos x$, as required.

2°. If $y=\sin u$, where $u=\varphi(x)$, then

$$\boxed{y'=(\sin u)'=\cos u\cdot u'} \qquad \text{(XII)}$$

Example. Find the derivative of the function $y=\sin(2-3x)$.

Solution. By formula (XII):

$$y'=\cos(2-3x)\cdot(2-3x)'=-3\cdot\cos(2-3x).$$

3°. *The function $\cos x$ has, at any point x, a derivative equal to $-\sin x$*, i.e., $(\cos x)'=-\sin x$.

Proof. It is known that $\cos x=\sin\left(\frac{\pi}{2}-x\right)$. Therefore,

$$(\cos x)'=\left[\sin\left(\frac{\pi}{2}-x\right)\right]'=\cos\left(\frac{\pi}{2}-x\right)\cdot\left(\frac{\pi}{2}-x\right)'=$$
$$=\cos\left(\frac{\pi}{2}-x\right)(-1)=-\sin x,$$

since $\left(\frac{\pi}{2}-x\right)'=-1$ and $\cos\left(\frac{\pi}{2}-x\right)=\sin x$, as required.

4°. If $y=\cos u$, where $u=\varphi(x)$, then

$$\boxed{y'=(\cos u)'=-\sin u\cdot u'} \qquad \text{(XIII)}$$

5°. *The function $\tan x$ has, at any point x, a derivative equal to* $\dfrac{1}{\cos^2 x}$, i.e., $(\tan x)'=\dfrac{1}{\cos^2 x}$.

Proof. Since $\tan x = \frac{\sin x}{\cos x}$, we get, by using formula (VII),

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \cdot \cos x - (-\sin x) \cdot \sin x}{\cos^2 x} =$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x},$$

as required.

6°. If $y = \tan u$, where $u = \varphi(x)$, then

$$\boxed{y' = (\tan u)' = \frac{1}{\cos^2 u} \cdot u'} \qquad \text{(XIV)}$$

7°. *The function cot x has, at any point x, a derivative equal to* $-\frac{1}{\sin^2 x}$, i.e., $(\cot x)' = -\frac{1}{\sin^2 x}$.

Proof. Since $\cot x = \frac{\cos x}{\sin x}$, we get, by using formula (VII),

$$(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x} =$$
$$= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x},$$

as required.

8°. If $y = \cot u$, where $u = \varphi(x)$,

$$\boxed{y' = (\cot u)' = -\frac{1}{\sin^2 u} \cdot u'} \qquad \text{(XV)}$$

9°. Examples. 1. Find the derivative of the function $y = \cot \frac{2}{x}$.

Solution. Using formula (XV) we get

$$y' = -\frac{1}{\sin^2 \frac{2}{x}} \cdot \left(\frac{2}{x}\right)' = -\frac{1}{\sin^2 \frac{2}{x}} \cdot \left(-\frac{2}{x^2}\right) = \frac{2}{x^2 \cdot \sin^2 \frac{2}{x}}.$$

2. Find the derivative of the function $y = \frac{1}{4} \sin^4 2x$.

Solution. The given function is a power function and may be written as $y = \frac{1}{4} (\sin 2x)^4$. Therefore, differentiating by formula (XI) we get

$$y' = \frac{1}{4} \cdot 4 (\sin 2x)^3 (\sin 2x)'.$$

We find the derivative of $\sin 2x$ by formula (XII):

$$y' = (\sin 2x)^3 \cdot \cos 2x \cdot (2x)' = 2 \sin^3 2x \cdot \cos 2x = \sin^2 2x \cdot \sin 4x.$$

3. Find the derivative of the function $y = \sqrt{1 - \tan^2 x}$.

Solution. The given function is a power function and may be written as

$$y=[1-(\tan x)^2]^{\frac{1}{2}}.$$

Then, by formula (XI),

$$y'=\frac{1}{2}[1-(\tan x)^2]^{-\frac{1}{2}}\cdot[1-(\tan x)^2]'=\frac{1}{2\sqrt{1-\tan^2 x}}\cdot(-2\tan x)\cdot(\tan x)'=$$

$$=-\frac{\tan x}{\sqrt{1-\tan^2 x\cdot\cos^2 x}}=-\frac{\tan x}{\cos x\sqrt{\cos 2x}}.$$

Sec. 91. Two Systems of Logarithms. The Number e. Changing from One System of Logarithms to the Other

1°. The function $y=\log_a x$ is called a logarithmic function. y is also called the logarithm of the number x to the base a. The other conditions are: $a>0$, $a\neq 1$ and number $x>0$. If the base $a=10$, the logarithm is called a decimal logarithm and is denoted by the symbol $\log x$ without the base indicated:

$$\log_{10} x=\log x.$$

Because of their simplicity decimal logarithms are very convenient to use in practical calculations. But in theoretical discussions *natural* logarithms are more convenient and are widely used. The base of natural logarithms is a number equal to the limit of the expression

$$\left(1+\frac{1}{m}\right)^m$$

when $|m|\to\infty$.

Academician Leonard Euler (1707-1783) of St. Petersburg investigated the formula (found by Daniel Bernoulli) for calculating this limit (which he denoted by e):

$$\lim_{|m|\to\infty}\left(1+\frac{1}{m}\right)^m=e=1+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\frac{1}{1\cdot 2\cdot 3\cdot 4}+\ldots \quad (1)$$

Figure 104 shows the graph of the functions $y=\left(1+\frac{1}{x}\right)^x$. To every value of x there corresponds a real number y only when the base $\left(1+\frac{1}{x}\right)$ is positive, which occurs only when $x>0$ or $x<-1$.

The number e is irrational, i.e., it is expressed by a nonterminating nonrepeating decimal fraction, and can be found to any desired degree of accuracy by formula (1). The sum

$$1+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\frac{1}{1\cdot 2\cdot 3\cdot 4}+\ldots+\frac{1}{1\cdot 2\cdot 3\ldots k}\approx e;$$

here, the error does not exceed $\frac{1}{k \cdot 1 \cdot 2 \cdot 3 \cdot 4 \ldots k}$,

$$e = 2.718281828459045\ldots.$$

Definition. *If the base of the logarithm is e, then the logarithm is a natural logarithm and is denoted by* ln *without the basal index*:

$$\log_e x = \ln x.$$

2°. *Changing from one system of logarithms to another.* Let us take the problem of finding the logarithm of a number to the

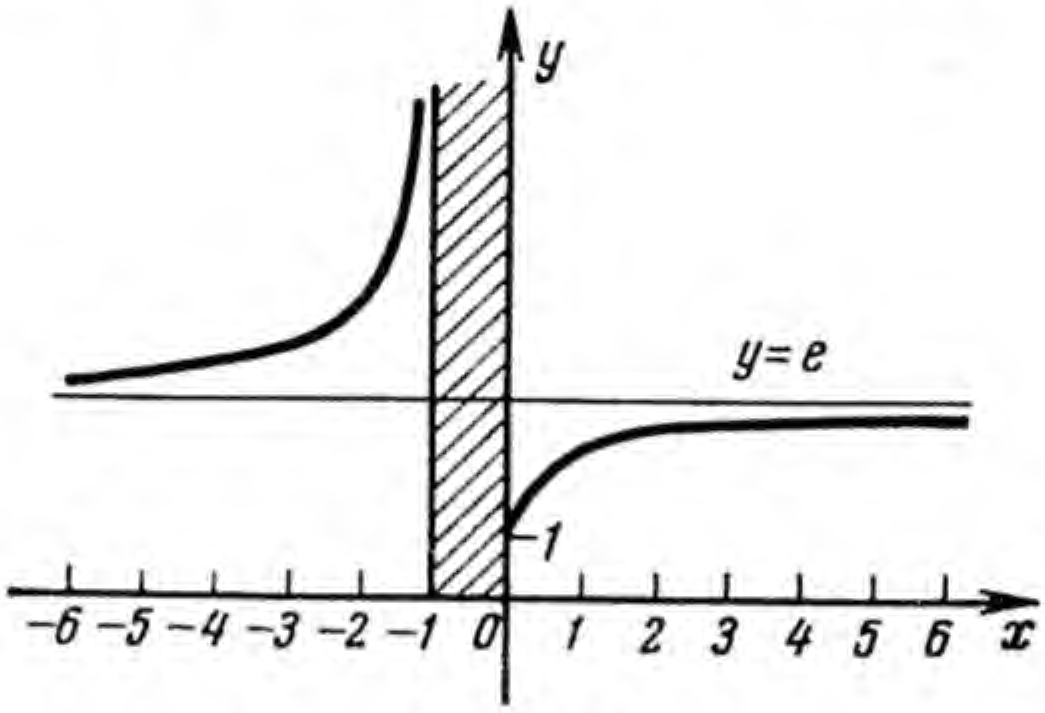

Fig. 104.

base a if the logarithm of the number to the base b is known. Denoting the number by N, we can write (by the definition of a logarithm)

$$N = a^{\log_a N} \quad \text{and} \quad N = b^{\log_b N}.$$

Whence

$$a^{\log_a N} = b^{\log_b N}.$$

Taking logarithms of both sides in the known system of logarithms, i.e., to the base b, we get

$$\log_a N \cdot \log_b a = \log_b N \cdot \log_b b = \log_b N,$$

since $\log_b b = 1$. Whence

$$\boxed{\log_a N = \log_b N \cdot \frac{1}{\log_b a}} \qquad \text{(A)}$$

i.e., *to obtain a logarithm to a new base it is sufficient to multiply the known logarithm by the inverse of the logarithm of the new base to the original base.*

The constant factor $\frac{1}{\log_b a}$ is called the modulus for a change of base from b to a.

Putting $a=e$ and $b=10$ in formula (A) we get the formula for changing from decimal to natural logarithms:

$$\boxed{\ln N = \log N \cdot \frac{1}{\log e}} \tag{1}$$

and putting $a=10$ and $b=e$ we get the formula for changing from natural into decimal logarithms:

$$\boxed{\log N = \ln N \cdot \frac{1}{\ln 10}} \tag{2}$$

The moduli have the following values:

$$\frac{1}{\log e} = 2.3025850929\ldots \quad \text{and} \quad \frac{1}{\ln 10} = 0.4342944819\ldots$$

3°. By formula (A):

$$\log_a N = \log_b N \cdot \frac{1}{\log_b a},$$

$$\log_b N = \log_a N \cdot \frac{1}{\log_a b}.$$

Multiplying these equations together,

$$\log_a N \cdot \log_b N = \log_b N \cdot \log_a N \cdot \frac{1}{\log_b a \cdot \log_a b},$$

and eliminating $\log_a N \cdot \log_b N$, we find that

$$\boxed{\log_a b \cdot \log_b a = 1} \tag{B}$$

In other words, the moduli $\log_a b$ and $\log_b a$ are reciprocals.

4°. Formulas (1) and (2) can be written as

$$\ln N = \log N \cdot \ln 10$$

and

$$\log N = \ln N \cdot \log e.$$

Sec. 92. The Derivative of a Logarithm

1°. $\ln x$ *has, at any point* x $(0 < x < +\infty)$, *a derivative equal to* $\frac{1}{x}$, i.e., $(\ln x)' = \frac{1}{x}$.

Proof. Putting $y = \ln x$, we have

$$\Delta y = \ln (x + \Delta x) - \ln x$$

or

$$\Delta y = \ln \frac{x + \Delta x}{x}$$

or

$$\Delta y=\ln\left(1+\frac{\Delta x}{x}\right).$$

To find the ratio $\frac{\Delta y}{\Delta x}$ we multiply the right side of the above equation by $\frac{1}{\Delta x}$:

$$\frac{\Delta y}{\Delta x}=\frac{1}{\Delta x}\cdot\ln\left(1+\frac{\Delta x}{x}\right).$$

Put $\frac{\Delta x}{x}=\frac{1}{m}$. Then, $\frac{1}{\Delta x}=\frac{m}{x}=\frac{1}{x}\cdot m$,

$$\frac{\Delta y}{\Delta x}=\frac{1}{x}\cdot m\cdot\ln\left(1+\frac{1}{m}\right).$$

Since

$$m\cdot\ln\left(1+\frac{1}{m}\right)=\ln\left(1+\frac{1}{m}\right)^m,$$

we have

$$\frac{\Delta y}{\Delta x}=\frac{1}{x}\cdot\ln\left(1+\frac{1}{m}\right)^m.$$

On the right-hand side of the above equation, m is the variable (x being constant), and $|m|=\left|\frac{x}{\Delta x}\right|\to\infty$ if $\Delta x\to 0$.

Let us find the limit of $\frac{\Delta y}{\Delta x}$ when $\Delta x\to 0$ and $|m|\to\infty$.

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{1}{x}\cdot\lim_{|m|\to\infty}\ln\left(1+\frac{1}{m}\right)^m,$$

or, in virtue of the continuity of logarithmic functions (Sec. 72, 1°):

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{1}{x}\cdot\ln\lim_{|m|\to\infty}\left(1+\frac{1}{m}\right)^m.$$

But

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=(\ln x)',\quad\text{and}\quad\lim_{|m|\to\infty}\left(1+\frac{1}{m}\right)^m=e.$$

Hence

$$(\ln x)'=\frac{1}{x}\cdot\ln e,$$

or, since $\ln e=\log_e e=1$, $(\ln x)'=\frac{1}{x}$, thus proving the proposition.

2°. If $y=\ln u$, where $u=\varphi(x)$, then

$$\boxed{y'=(\ln u)'=\frac{1}{u}\cdot u'}\qquad\textbf{(XVI)}$$

3°. *The function* $\log_a x$ *has at any point* x $(0<x<+\infty)$ *a derivative equal to* $\frac{1}{x\cdot\ln a}$, i.e., $(\log_a x)'=\frac{1}{x\cdot\ln a}$.

Proof. We have, by putting $b=e$ and $N=x$ in formula (A) for change of base,

$$\log_a x=\ln x\cdot\frac{1}{\ln a}.$$

Since $\frac{1}{\ln a}$ is a constant factor, formula (III) gives us

$$(\log_a x)'=(\ln x)'\cdot\frac{1}{\ln a},$$

or $(\log_a x)'=\frac{1}{x\cdot\ln a}$, thus proving the proposition.

4°. If $y=\log_a u$, where $u=\varphi(x)$, then

$$\boxed{y'=(\log_a u)'=\frac{1}{u\cdot\ln a}\cdot u'} \qquad \text{(XVII)}$$

5°. **Examples. 1.** Find the derivative of the function $y=\log_a(x^2+4)$.

Solution. Applying formula (XVII), we get

$$y'=\frac{1}{(x^2+4)\cdot\ln a}\cdot(x^2+4)'=\frac{2x}{(x^2+4)\cdot\ln a}.$$

2. Find the derivative of the function $y=\ln(x^2+4)^2$.

Solution. When differentiating a logarithmic function it is generally useful to take logarithms first (if this is possible of course). This simplifies finding the derivative. Taking the logarithm of $(x^2+4)^2$, we get

$$y=2\cdot\ln(x^2+4).$$

Applying formulas (III) and (XVI), we get

$$y'=2\cdot\frac{1}{x^2+4}\cdot(x^2+4)'=\frac{4x}{x^2+4}.$$

3. Find derivative of $y=\ln\sqrt{\frac{1+\sin x}{1-\sin x}}$.

Solution. Taking the logarithm of the root and the fraction, we get

$$y=\frac{1}{2}\cdot\ln(1+\sin x)-\frac{1}{2}\ln(1-\sin x).$$

Differentiating the difference and applying formulas (III), (XVI) and (XII), we find

$$y'=\frac{1}{2}\cdot\frac{1}{1+\sin x}\cdot\cos x-\frac{1}{2}\cdot\frac{1}{1-\sin x}\cdot(-\cos x)=$$

$$=\frac{\cos x}{2}\left(\frac{1}{1+\sin x}+\frac{1}{1-\sin x}\right)=\frac{\cos x}{2}\cdot\frac{2}{1-\sin^2 x}=\frac{\cos x}{\cos^2 x}=\frac{1}{\cos x}.$$

The answer turns out rather simple: $\frac{1}{\cos x}$. Taking a second look at the original function, we discover that it may be simplified by using the

formulas of trigonometry. Indeed,

$$\frac{1+\sin x}{1-\sin x}=\frac{1+\cos\left(\frac{\pi}{2}-x\right)}{1-\cos\left(\frac{\pi}{2}-x\right)}=\cot^2\frac{1}{2}\left(\frac{\pi}{2}-x\right)=\cot^2\left(\frac{\pi}{4}-\frac{x}{2}\right).$$

$$y=\ln\sqrt{\frac{1+\sin x}{1-\sin x}}=\ln\cot\left(\frac{\pi}{4}-\frac{x}{2}\right).$$

We find the derivative:

$$y'=\left[\ln\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]'=\frac{1}{\cot\left(\frac{\pi}{2}-\frac{x}{2}\right)}\cdot\left(-\frac{1}{\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right)\times$$

$$\times\left(-\frac{1}{2}\right)=\frac{1}{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cdot\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}=\frac{1}{\sin 2\left(\frac{\pi}{4}-\frac{x}{2}\right)}=$$

$$=\frac{1}{\sin\left(\frac{\pi}{2}-x\right)}=\frac{1}{\cos x}.$$

Did we save time and energy in first simplifying the function to be differentiated? Evidently not. The solution became longer and called for the application of a number of trigonometrical formulas, while the first method required only one formula. Thus, it is not always true that prelimi-nary simplification of a given function leads to quicker results. Therefore, this is not usually done when differentiating.

Sec. 93. Monotonic Functions

Definition. 1. *The function $f(x)$ is called an increasing function if, for any two different values of the argument x, to the greater value of x there always corresponds a greater value of the function, i.e.,*

$$\text{if } x_1<x_2 \quad \text{and} \quad f(x_1)<f(x_2).$$

2. *The function $f(x)$ is called a decreasing function of x if, for any two different values of the argument x, to the greater value of the argument there corresponds a smaller value of the function, i.e.,*

$$\text{if } x_1<x_2 \quad \text{and} \quad f(x_1)>f(x_2).$$

Examples. 1. The exponential function $y=a^x$, where $a>1$, is an increasing function in the entire domain of its definition:

$$-\infty<x<+\infty.$$

2. The function $\sin x$ is an increasing function in some intervals and a decreasing function in other intervals. For example, in the interval $-\frac{\pi}{2}<x<+\frac{\pi}{2}$ $\sin x$ is an increasing function, and in the interval $\frac{\pi}{2}<x<\frac{3\pi}{2}$ it is a decreasing function.

3. *The function $f(x)$ is called a nondecreasing function if for all values of x_1 and x_2, when $x_1 < x_2$, $f(x_1) \leqslant f(x_2)$.*

Similarly, the *function $f(x)$ is said to be a nonincreasing one if for $x_1 < x_2$, $f(x_1) \geqslant f(x_2)$.*

Thus a nondecreasing function differs from an increasing function in that, in the former, two values of the function (at different values of the argument) may be equal, while this is impossible in the case of an increasing function.

4. *Nondecreasing and nonincreasing functions are called monotonic functions; and increasing and decreasing functions are called strictly monotonic functions.*

Sec. 94. The Derivative of an Inverse Function

1°. The equation $y = f(x)$ defines y as a (single-valued) function of x. If it also permits defining x as a single-valued function of y, $x = \overline{f}(y)$, then the first function, $y = f(x)$, is called a *direct function*, and the second one, $x = \overline{f}(y)$, is called an *inverse function*.

Denoting the inverse function in the ordinary way, i.e., not as $x = \overline{f}(y)$ but as $y = \overline{f}(x)$, we obtain for the direct function: $x = f(y)$.

2°. **Theorem.** *If: 1) $x = f(y)$ is a strictly monotonic function and 2) has at point y a derivative $f'(y)$ not equal to zero, then:*

1) *the inverse function $y = \overline{f}(x)$ has a derivative at point x, corresponding to point y; and*

2) *the derivative, with respect to x, of the inverse function $\overline{f}(x)$ is equal to unity divided by the derivative of the direct function $f(y)$ with respect to y,* i.e.,

$$\boxed{\overline{f}'_x(x) = \frac{1}{f'_y(y)} \quad \text{or} \quad y'_x = \frac{1}{x'_y}} \qquad \text{(XVIII)}$$

Proof. By giving y an increment $\Delta y \neq 0$, we obtain in the function $x = f(y)$ an increment Δx such that $x + \Delta x = f(y + \Delta y)$.

In view of the strict monotony of the function $x = f(y)$

$$x + \Delta x \neq x, \text{ i.e., } \Delta x \neq 0.$$

By the definition of a derivative we have

$$y'_x = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\frac{\Delta x}{\Delta y}}\,.$$

Now, since the function $x = f(y)$ is continuous, $\Delta x \to 0$ if $\Delta y \to 0$ (Sec. 70); hence (Sec. 78) $\Delta x \to 0$ can be replaced by

$\Delta y \to 0$. Thus

$$y'_x = \lim_{\Delta x \to 0} \frac{1}{\frac{\Delta x}{\Delta y}} = \frac{1}{\lim\limits_{\Delta y \to 0} \frac{\Delta x}{\Delta y}} = \frac{1}{x'_y}.$$

Sec. 95. The Derivative of an Exponential Function

1°. We can treat the exponential function $y=a^x$ (where $a>0$, $-\infty<x<+\infty$, $0<y<+\infty$) as the converse of the logarithmic function:

$$x=\log_a y.$$

The function $x=\log_a y$ is strictly monotonic (increasing when $a>1$ and decreasing when $a<1$) and has a derivative at y. From the foregoing, the inverse function $y=a^x$ has a derivative at x and is found by formula (XVIII):

$$(a^x)'_x = \frac{1}{(\log_a y)'_y} = \frac{1}{\frac{1}{y \cdot \ln a}} = y \cdot \ln a.$$

Replacing y by a^x we have

$$\boxed{(a^x)' = a^x \cdot \ln a} \qquad \text{(XIXa)}$$

2°. If base $a=e$, i.e., if $y=e^x$, then

$$\boxed{(e^x)' = e^x} \qquad \text{(XXa)}$$

since $\ln a = \ln e = 1$.

3°. If the exponential function is a function of a function: $y=a^u$ or $y=e^u$, where $u=\varphi(x)$, then

$$\boxed{y' = (a^u)' = a^u \cdot u' \cdot \ln a} \qquad \text{(XIX)}$$

$$\boxed{y' = (e^u)' = e^u \cdot u'} \qquad \text{(XX)}$$

4°. The function $y=a^x$ can always be represented in the form of an exponential function with base e as follows: from equation $y=a^x$ it follows that

$$\ln y = x \cdot \ln a.$$

Hence

$$y=e^{x \ln a}.$$

Putting $\ln a = k$, we get the equation of the exponential function $y=a^x$ in the form

$$y=e^{kx}.$$

Functions $y=e^{kx}$ and $y=c \cdot e^{kx}$, where k is a constant, are frequently met with in practice.

Sec. 96. The Derivative of Any Power

1°. Formula (II) for the derivative of a power function was hitherto not proved for fractional and irrational exponents. We shall now proceed to give the proof. First let us note that powers with fractional and irrational exponents are here considered only for the case of positive bases.

In the equation $y = x^n$, $x > 0$ and n is a fractional or irrational number. Taking the logarithms of $y = x^n$ to the base e, we get

$$\ln y = n \ln x.$$

Whence

$$y = e^{n \ln x}.$$

Since $e^{n \ln x}$ has a derivative, x^n must also have a derivative. By formula (XX),

$$y' = (e^{n \ln x})' = e^{n \ln x} \cdot n \cdot \frac{1}{x} = x^n \cdot n \cdot \frac{1}{x} = n \cdot x^{n-1}.$$

Consequently, $(x^n)' = n \cdot x^{n-1}$ thus proving the theorem.

2°. *Functions of the type u^v, where the base u and the exponent v are separate functions of x, are called composite exponential functions.*

Assuming $u > 0$ and the functions u and v to be differentiable with respect to x, we shall show that u^v has a derivative.

Taking the logarithms of $y = u^v$ to the base e, we get

$$\ln y = v \cdot \ln u;$$

$$y = e^{v \cdot \ln u}, \quad \text{or} \quad u^v = e^{v \cdot \ln u}.$$

Since $e^{v \cdot \ln u}$ has a derivative, u^v also has a derivative, which is found by formula (XX).

3°. **Examples. 1.** Find the derivative of $y = x^x$.

Solution. $x^x = e^{x \cdot \ln x}$. Hence by formula (XX):

$$y' = (e^{x \cdot \ln x})' = e^{x \cdot \ln x} \cdot (x \cdot \ln x)' = e^{x \cdot \ln x} \cdot (\ln x + 1) = x^x \cdot (\ln x + 1).$$

2. Find the derivative of $y = (\cos x)^{\sin x}$.

Solution.

$$(\cos x)^{\sin x} = e^{\sin x \cdot \ln \cos x},$$

$$y' = (e^{\sin x \cdot \ln \cos x})' = e^{\sin x \cdot \ln \cos x} \cdot (\sin x \cdot \ln \cos x)' =$$

$$= e^{\sin x \ \ln \cos x} \cdot \left(\cos x \cdot \ln \cos x - \frac{\sin x}{\cos x} \cdot \sin x \right) =$$

$$= (\cos x)^{\sin \cdot x} \cdot (\cos x \cdot \ln \cos x - \sin x \cdot \tan x).$$

Sec. 97. Derivatives of Inverse Trigonometric Functions

1°. The functions $\sin y$ and $\tan y$ increase in the interval $-\frac{\pi}{2} < y < +\frac{\pi}{2}$, and $\cos y$ and $\cot y$ decrease in the interval $0 < y < \pi$.

i.e., these functions are strictly monotonic and have derivatives at every point of these intervals. Therefore their inverse functions. $\arcsin x$ and $\arccos x$ in the interval $-1<x<+1$ and $\arctan x$ and $\operatorname{arc\,cot} x$ in the interval $-\infty<x<+\infty$, have derivatives (Sec. 94).

2°. If $y=\arcsin x$, then $x=\sin y$. By formula (XVIII),

$$(\arcsin x)'_x=\frac{1}{(\sin y)'_y}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-\sin^2 y}}\text{*}.$$

Since $\sin y=x$, by replacing $\sin^2 y$ by x^2 we obtain

$$\boxed{(\arcsin x)'=\frac{1}{\sqrt{1-x^2}}} \tag{XXIa}$$

3°. If $y=\arccos x$, $x=\cos y$. By formula (XVIII),

$$(\arccos x)'_x=\frac{1}{(\cos y)'_y}=-\frac{1}{\sin y}=-\frac{1}{\sqrt{1-\cos^2 y}}\text{**}$$

or

$$\boxed{(\arccos x)'=-\frac{1}{\sqrt{1-x^2}}} \tag{XXIIa}$$

4°. If $y=\arctan x$, $x=\tan y$. By formula (XVIII),

$$(\arctan x)'_x=\frac{1}{(\tan y)'_y}=\frac{1}{\sec^2 y}=\frac{1}{1+\tan^2 y}.$$

Since $\tan y=x$, we get, by replacing $\tan^2 y$ by x^2,

$$\boxed{(\arctan x)'=\frac{1}{1+x^2}} \tag{XXIIIa}$$

5°. If $y=\operatorname{arc\,cot} x$, $x=\cot y$. By formula (XVIII),

$$(\operatorname{arc\,cot} x)'_x=\frac{1}{(\cot y)'_y}=-\frac{1}{\operatorname{cosec}^2 y}=-\frac{1}{1+\cot^2 y}$$

or

$$\boxed{(\operatorname{arc\,cot} x)'=-\frac{1}{1+x^2}} \tag{XXIVa}$$

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*) The plus sign in front of the radical is because $\cos y>0$ since $-\frac{\pi}{2}<y<+\frac{\pi}{2}$.

**) $\sin y>0$ because $0<y<\pi$.

6°. If the inverse trigonometric function is a function of u, where $u=\varphi(x)$, then

$(\operatorname{arc}\sin u)'=\dfrac{1}{\sqrt{1-u^2}}\cdot u'$	(XXI)
$(\operatorname{arc}\cos u)'=-\dfrac{1}{\sqrt{1-u^2}}\cdot u'$	(XXII)
$(\operatorname{arc}\tan u)'=\dfrac{1}{1+u^2}\cdot u'$	(XXIII)
$(\operatorname{arc}\cot u)'=-\dfrac{1}{1+u^2}\cdot u'$	(XXIV)

7°. **Examples.** 1. Find the derivative of $y=\operatorname{arc}\sin ax$.
Solution. By formula (XXI),

$$y'=\frac{1}{\sqrt{1-(ax)^2}}\cdot(ax)'=\frac{a}{\sqrt{1-a^2x^2}}.$$

2. Find the derivative of $y=\operatorname{arc}\tan(2x^2-7)$.
Solution. By formula (XXIII),

$$y'=\frac{1}{1+(2x^2-7)^2}\cdot(2x^2-7)'=\frac{4x}{1+4x^4-28x^2+49}=\frac{2x}{2x^4-14x^2+25}.$$

Sec. 98. Derivatives of Second and Higher Orders

The derivative $f'(x)$ of the function $f(x)$ is itself a function of the independent variable x. This derivative may be a differentiable function and hence its derivative can in turn be found.

Example. The given function is $y=x^4$; the derivative of the given function is $(x^4)'=4x^3$; the derivative of this derivative is $(4x^3)'=12x^2$.

The derivative of the given function is called the first derivative, or a derivative of the first order; the derivative of the first derivative is called the second derivative, or a derivative of the second order. The notation is y'', $f''(x)$ or $y^{(2)}$, $f^{(2)}(x)$, or $\frac{d^2y}{dx^2}$, $\frac{d^2f(x)}{dx^2}$. The derivative of the second derivative is called the third derivative, or a derivative of the third order, and is denoted by y''', $f'''(x)$, or $y^{(3)}$, $f^{(3)}(x)$, or $\frac{d^3y}{dx^3}$, $\frac{d^3f(x)}{dx^3}$, etc.

The process of finding derivatives one after the other is called *successive differentiation.*

Examples:

1. $y=x^4;\quad y'=4x^3;\quad y''=12x^2;\quad y'''=24x;\quad y^{\mathrm{IV}}=24;\quad y^{\mathrm{V}}=0$

2. $y=\ln x;\quad y'=\frac{1}{x}=x^{-1};\quad y''=-1\cdot x^{-2}=-\frac{1}{x^2};$

$$y'''=+1\cdot2\cdot x^{-3}=\frac{1\cdot2}{x^3};\quad y^{\mathrm{IV}}=-1\cdot2\cdot3x^{-4}=-\frac{1\cdot2\cdot3}{x^4},\ \text{etc.}$$

CHAPTER VIII

STUDYING FUNCTIONS WITH THE AID OF THEIR DERIVATIVES

Sec. 99. *How to Determine Whether a Function Increases, Decreases or Is Constant**

Let $y=f(x)$ be a definite and differentiable function at every point of the interval

$$a \leqslant x \leqslant b.$$

1°. It is known from Sec. 80 that a constant function has a zero derivative at every point of the interval. In advanced courses of analysis the converse is proved: *that the function $f(x)$ is constant over the interval $[a, b]$ if at every point of the interval its derivative $f'(x)$ is zero.*

Let us show this graphically (Fig. 105). If $f'(x)=0$ at every point of the interval $[a, b]$, the tangent to the graph of $y=f(x)$

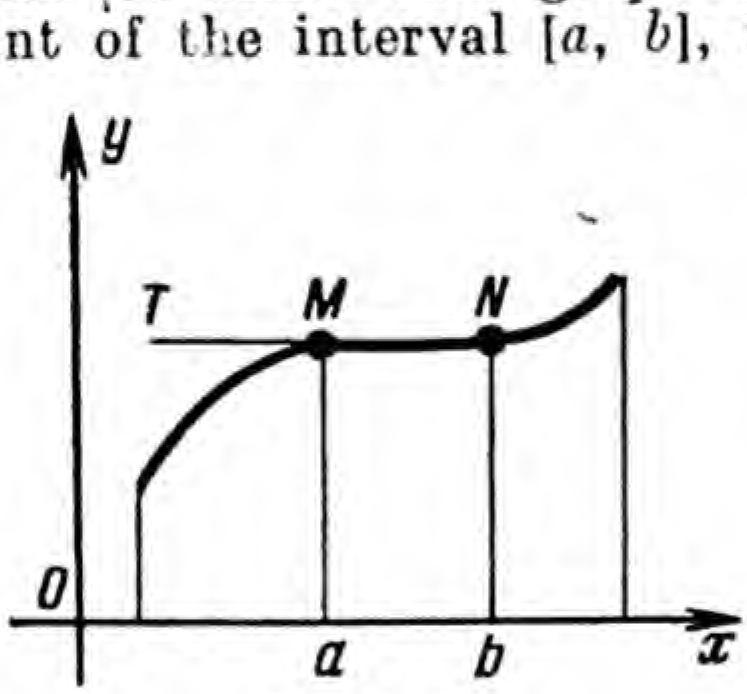

Fig. 105.

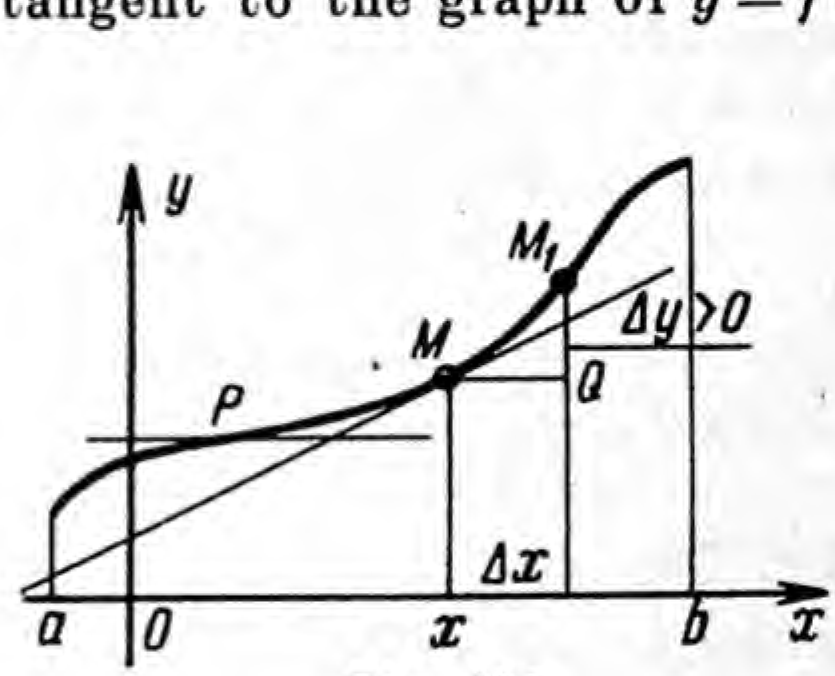

Fig. 106.

is parallel to the x-axis at every point $x\,(a \leqslant x \leqslant b)$. As the value of x varies, point M of the graph (which is the point of contact of the tangent) moves to the right but retains the direction of the tangent at M since the tangent does not change its direction over the interval $a \leqslant x \leqslant b$. As a result, the curve of $y=f(x)$ over this interval is obtained as a straight line MN parallel to the x-axis, and the value of the function, $f(a)$, remains constant.

* Reread Secs. 93 and 77 before beginning this chapter.

2°. If $y=f(x)$ is an increasing function over the interval $a<x<b$ (Fig. 106), then, as x increases, each successive value of y is greater than the preceding value; and, hence, for every given value of x, the increments Δx and Δy are both positive. The relation $\frac{\Delta y}{\Delta x}$ is also positive and remains so whenever Δx tends to zero. As a result (Sec. 55), its limiting value—the derivative $f'(x)$—is either positive or zero:

$$f'(x) \geqslant 0.$$

If $y=f(x)$ decreases over the interval $a<x<b$ (Fig. 107), every increase in x leads to a decrease in the value of y. In other words, for a positive increment Δx in any value of x, the increment Δy is negative. The ratio $\frac{\Delta y}{\Delta x}$ is always negative, and in the limit, as $\Delta x \to 0$, the value of the ratio is a negative number or zero.

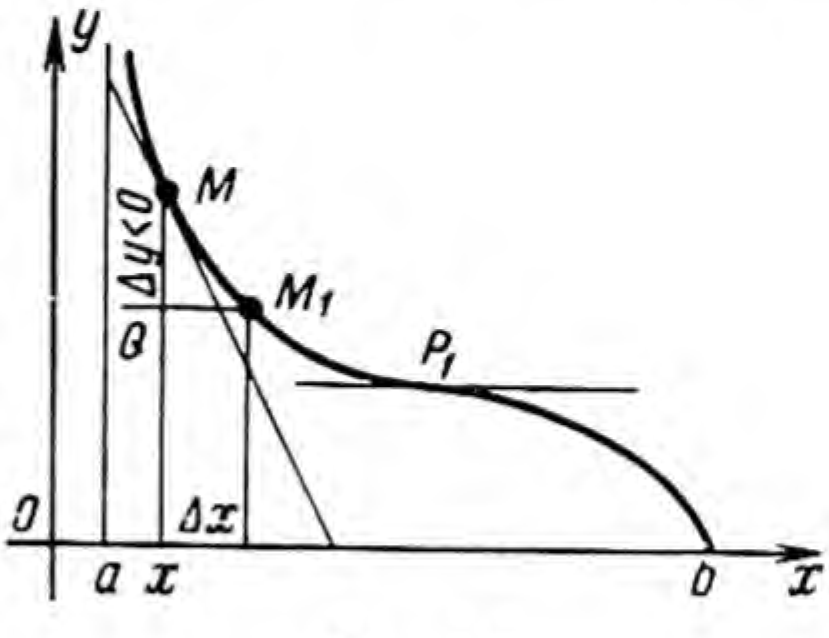

Fig. 107.

$$f'(x) \leqslant 0.$$

Since the value of the derivative $f'(x)$ is equal to the slope of the tangent to the graph of the function $y=f(x)$:

$$f'(x)=\tan\varphi,$$

and in the case of an increasing function $f'(x)=\tan\varphi \geqslant 0$, the tangent to the graph of the increasing function makes an acute angle with the x-axis or is parallel to it (Fig. 106). For a decreasing function, $f'(x)=\tan\varphi \leqslant 0$, and the tangent to the graph makes an obtuse angle with the x-axis or is parallel to it (Fig. 107).

In the interval $a<x<b$ of an increasing (or decreasing) function there is no interval $a_1 \leqslant x \leqslant b_1$ (where $a<a_1<b_1<b$), at all the points of which the derivative is zero, for if $f'(x)$ were zero over the interval $a_1 \leqslant x \leqslant b_1$, the function $f(x)$ would have one and the same value at all the points of this interval; i.e., it would not be an increasing (or decreasing) function.

The points of the graph of an increasing (or decreasing) function where the tangent is parallel to the x-axis are isolated points in the sense that their abscissas do not constitute an interval. Such are the points P and P_1 in Figs. 106 and 107.

3°. In advanced courses of analysis the following conditions are shown to be sufficient to determine whether a function is an increasing or decreasing one.

The function $f(x)$ increases (or decreases) in the interval $a<x<b$ if:

1) *the derivative* $f'(x)$ *is not negative (or not positive) in the interval* $a<x<b$,

$$f'(x)\geqslant 0 \ [\text{or } f'(x)\leqslant 0], \text{ and}$$

2) *no subinterval* $a_1\leqslant x\leqslant b_1$ $(a<a_1<b_1<b)$ *exists in the interval, at all the points of which the derivative* $f'(x)=0$.

4° **Example.** Determine the intervals over which the function $y=x^3-x^2-8x+2$ increases and decreases.

Solution. To make use of the conditions of an increasing or decreasing function, find the derivative of the function and determine the values of x for which the derivative is positive or negative:

$$y'=3x^2-2x-8.$$

Factorise the right-hand side of the equation, since it is far easier to conclude about the sign of a product from the signs of its factors than about the sign of a sum from the signs of its terms.

The roots of the trinomial are

$$x=\frac{1\pm\sqrt{1+24}}{3}=\frac{1\pm 5}{3}; \quad x_1=-\frac{4}{3}, \quad x_2=2.$$

Whence

$$y'=3\cdot\left(x+\frac{4}{3}\right)\cdot(x-2).$$

Factor $x+\frac{4}{3}$ is negative when $x<-\frac{4}{3}$ and positive when $x>-\frac{4}{3}$. Factor $x-2$ is negative when $x<2$ and positive when $x>2$. The product will be positive or negative depending on the position of x on the x-axis with respect to points $-\frac{4}{3}$ and 2. These points divide the x-axis into three intervals:

$$1)\ -\infty<x<-\frac{4}{3}, \qquad 2)\ -\frac{4}{3}<x<2, \qquad 3)\ 2<x<+\infty.$$

To determine the sign of the derivative in each interval, we tabulate the results:

Number of interval	Characteristic of interval	Sign of $x+\frac{4}{3}$	Sign of $x-2$	Sign of $f'(x)$	The function
1	$-\infty<x<-\frac{4}{3}$	$-$	$-$	$+$	increases
2	$-\frac{4}{3}<x<2$	$+$	$-$	$-$	decreases
3	$2<x<+\infty$	$+$	$+$	$+$	increases

Thus the function increases in the intervals $-\infty<x<-\frac{4}{3}$ and $2<x<+\infty$ and decreases in the interval $-\frac{4}{3}<x<2$. The graph of the function is shown in Fig. 108.

5°. The function $y=x^3$ (Fig. 109) has the derivative $y=3x^2$, which is positive for all values of x except $x=0$. When $x=0$, the derivative $y'=0$. The function $y=x^3$ increases in the interval $-\infty < x < +\infty$; $x=0$ is an

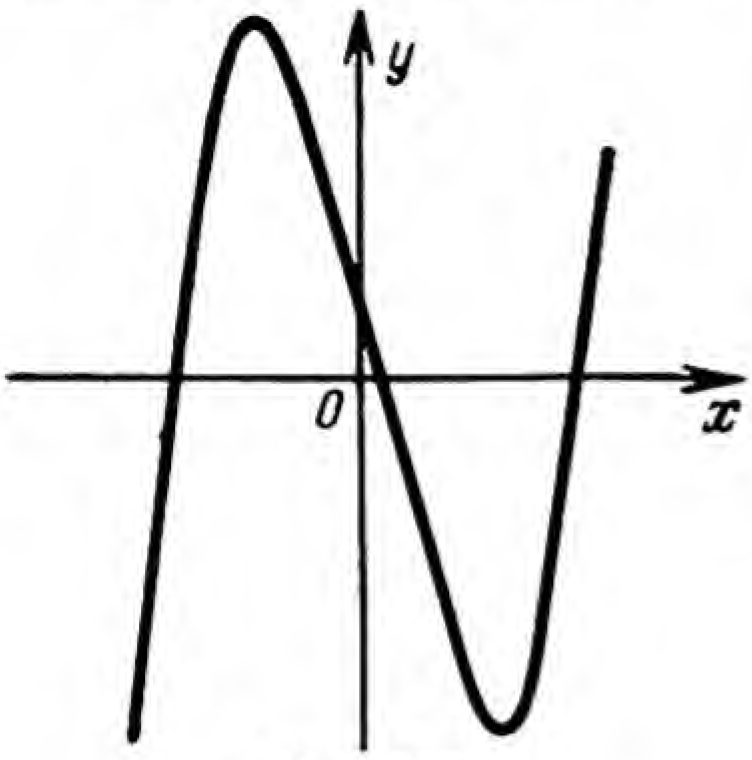

Fig. 108.

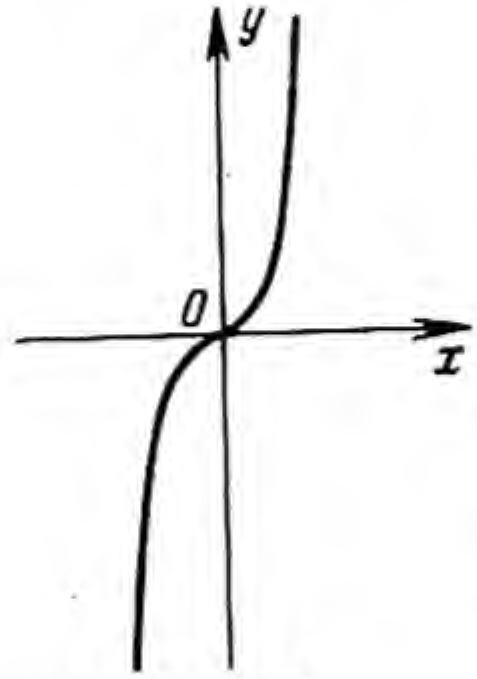

Fig. 109.

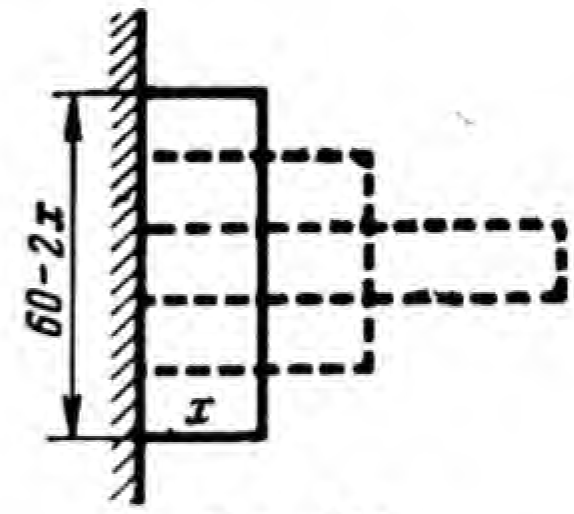

Fig. 110.

isolated (and the only) point for which the derivative is zero; here the function increases. Indeed, when $x=0$, $x^3=0$ and when $x<0$, $x^3<0$; lastly, when $x>0$, $x^3>0$.

Sec. 100. Extreme Value Problems

1°. It is required to fence off the largest possible rectangular area adjoining the wall of a house (Fig. 110). A wire net 60 metres in length is available. Find the dimensions of the rectangle.

Solution. Let the width of the rectangle be x metres (m) and its area y m^2. Then

$$y=(60-2x)\cdot x=60x-2x^2.$$

Since x and y cannot be negative, the factor $60-2x>0$ and $0<x<30$.

The area y is a function of x. Let us find the intervals of x-values for which y increases and decreases:

$$y'=60-4x.$$

For $x<15$, $y'>0$ and the function increases; for $x>15$, $y<0$ and the function decreases.

If the width	$x=$	0	5	10	15	20	25	30
Then the area	$y=$	0	250	400	450	400	250	0

The curve (Fig. 111) first rises from the origin 0 to M $(x=15)$ and then falls off. At $x=15$ the function is a maximum.

Hence the area of the plot is greater (maximum) if $x=15$ metres (width) and $60-2x=30$ metres (length).

2°. What should be the dimensions of a room of area 36 m² so that its perimeter is the smallest possible?

Solution. Let the length be x m. Then the width is $\frac{36}{x}$ m, and the perimeter

$$y=2\left(x+\frac{36}{x}\right)=2x+\frac{72}{x}.$$

Since the perimeter y is a function of the length x defined for all positive values of x,

$$0<x<+\infty.$$

Let us find the intervals in which it increases and decreases:

$$y'=2-\frac{72}{x^2}=\frac{2(x^2-36)}{x^2}=\frac{2(x-6)(x+6)}{x^2}.$$

The sign of the derivative is determined by the sign of the difference $x-6$. In the interval $0<x<6$, $y'<0$; and in the interval $6<x<+\infty$, $y'>0$. The perimeter decreases over the interval

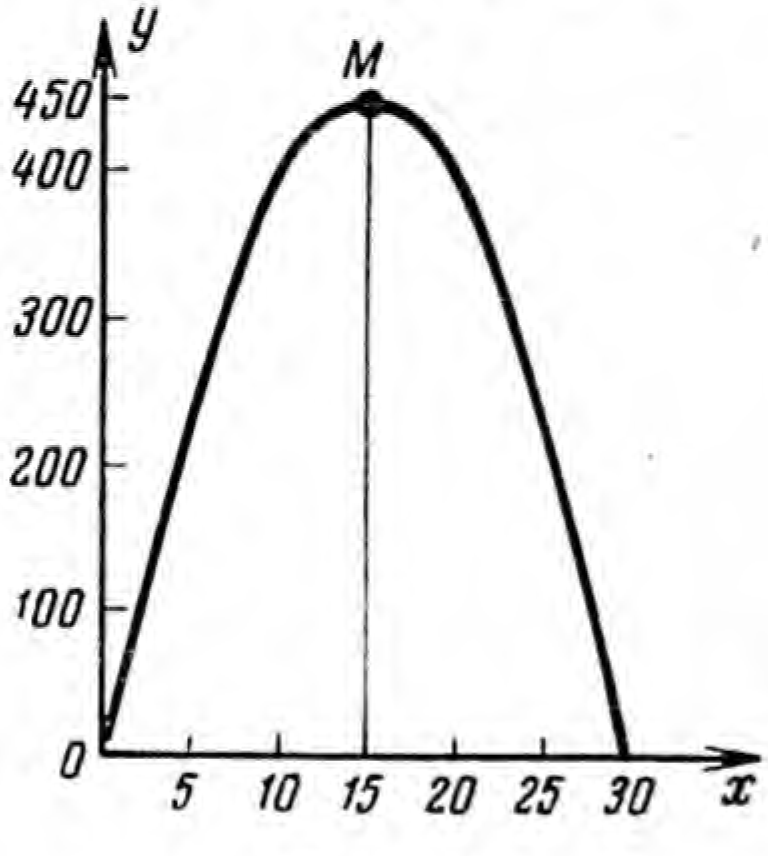

Fig. 111.

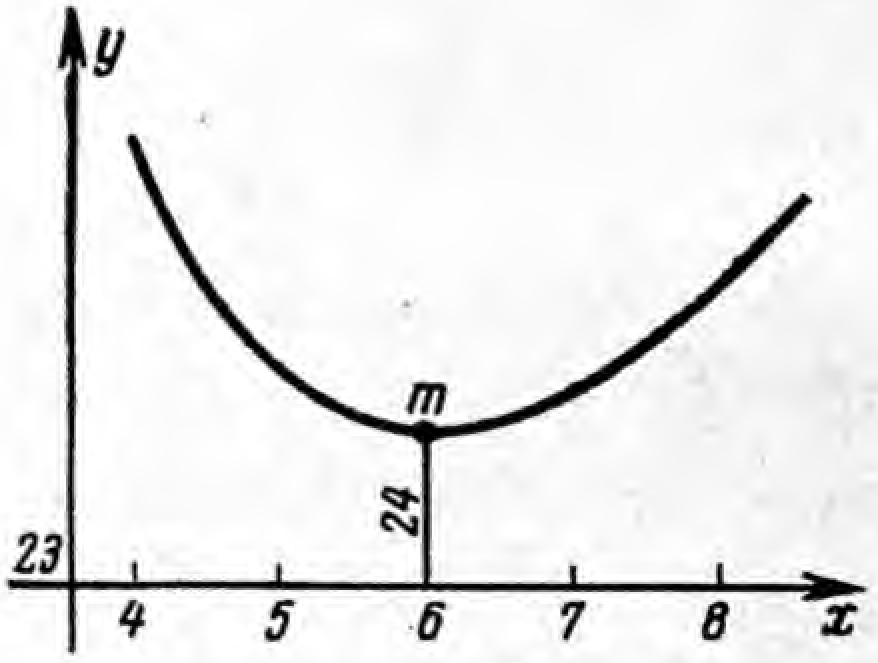

Fig. 112.

$0<x<6$ and increases over the interval $6<x<+\infty$. The curve in Fig. 112 is plotted from the following table:

If $x=$	$\to 0$	3	4	5	6	7	8	$\to\infty$
Then $y=$	$\to\infty$	30	26	24.4	24	24.3	25	$\to\infty$

Hence the perimeter of the rectangle is smallest (minimum) when the length is 6 metres and the width is $\frac{36}{6}=6$ metres, i.e., when the room is a square.

Sec. 101. Maximum and Minimum of a Function

It is very often important in engineering practice to know the greatest (or least) value of a quantity, and, as is clear from the foregoing examples, the problem reduces to finding the maximum and minimum of a function.

Definition. 1. *The function $f(x)$ is said to have a maximum at $x=c$ if its value at this point is greater than at any point within a certain neighbourhood of $x=c$.*

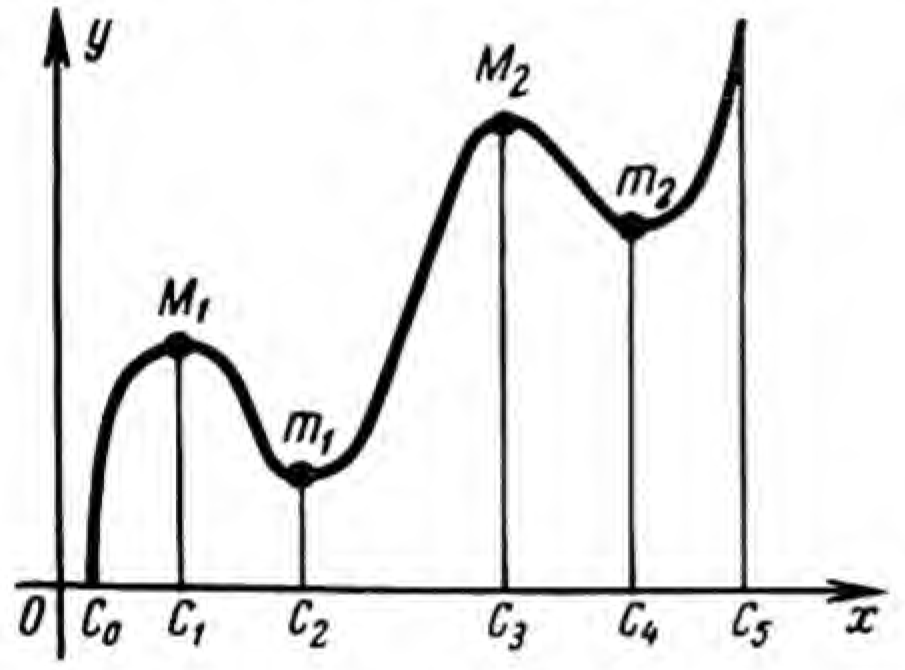

Fig. 113.

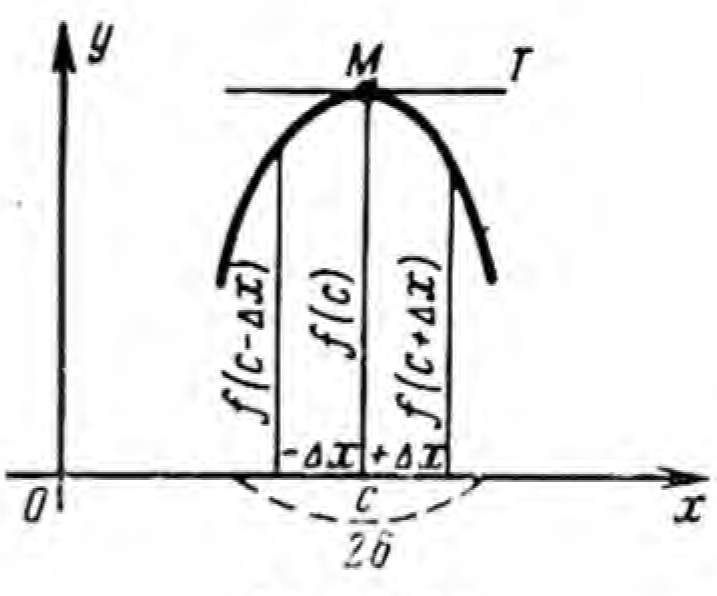

Fig. 114.

2. *The function $f(x)$ has a minimum at $x=c$ if its value at $x=c$ is smaller than at any point within a certain neighbourhood of $x=c$.*

The generic term for "maximum" and "minimum" is "extremum" or "extreme".

The value of the independent variable for which the function has a *maximum* (or *minimum*) is called the *maximum point* (or the *minimum point*) or the *extremum point*.

A function may have only a maximum, e.g., the function $y=60x-2x^2$ (Fig. 111), or only a minimum, e.g., the function $y=2x+\frac{72}{x}$ (Fig. 112) or a maximum and a minimum, e.g., $y=x^3-x^2-8x+2$ (Fig. 108). A function may also have several maxima and minima (Fig. 113); and in such a case the maxima alternate with the minima. A function may also be without any maximum or minimum. For example, $y=x^3$, $y=\cot x$, $y=a^x$ do not have either a maximum or a minimum since as x increases from $-\infty$ to $+\infty$, the first and third functions increase while the second function only decreases.

The maximum (minimum) of a function may not represent its greatest (least) value. Thus the function depicted in Fig. 113 has at c_5 a value greater than its maximum values c_1M_1 and c_3M_2, and at c_0 the value of the function is less than its minima c_2m_1 and c_4m_2. The minimum value c_4m_2 is greater than the maximum value c_1M_1. The maximum (minimum) of a function at a given point generally represents its greatest (least) value in comparison with the values of the function at points to the right and left of the particular extremum point *only in a sufficiently close neighbourhood.*

Sec. 102. A Test for Extremes

1°. **Theorem** (necessary condition). *If in the neighbourhood 2δ of the point $x=c$:*

1) *the function $f(x)$ is differentiable, and* 2) *$x=c$ is an extremum point of the function $f(x)$, then its derivative at c is zero, i.e., $f'(c)=0$.*

Proof. To be definite, let $x=c$ be a maximum point (Fig. 114). Let $c-\Delta x$ represent the values of the independent variable x in the left semineighbourhood of c, and $c+\Delta x$, the values in the right semineighbourhood, such that $0<\Delta x<\delta$. The value of the function $f(x)$ at c is $f(c)$. In the left semineighbourhood it is $f(c-\Delta x)$, in the right, $f(c+\Delta x)$. Thus the values of $f(x)$ within the neighbourhood 2δ of c represent a function of Δx, and the values $x=c\mp\Delta x$ approach c without bound if $\Delta x\to 0$.

By the definition of the maximum of a function,

$$f(c-\Delta x)<f(c) \quad \text{and} \quad f(c+\Delta x)<f(c).$$

Whence,

$$f(c-\Delta x)-f(c)<0 \quad \text{and} \quad f(c+\Delta x)-f(c)<0.$$

The left sides of the inequalities express an increment in the function at the point $x=c$ when the argument changes by $-\Delta x$ and $+\Delta x$, respectively. Let us write down the ratio of the increment of the function to the increment of the argument:

$$\frac{f(c-\Delta x)-f(c)}{-\Delta x}>0 \quad (1): \qquad \frac{f(c+\Delta x)-f(c)}{+\Delta x}<0. \qquad (2)$$

Both the relations (1) and (2) have the same limiting value when $\Delta x\to 0$ since it is given that $f(x)$ has a definite derivative at the point c:

$$\lim_{-\Delta x\to 0}\frac{f(c-\Delta x)-f(c)}{-\Delta x}=f'(c) \quad \text{and} \quad \lim_{+\Delta x\to 0}\frac{f(c+\Delta x)-f(c)}{+\Delta x}=f(c).$$

It follows from inequality (1) (see Sec. 55) that $f'(c)$ is either positive or zero. Inequality (2) shows that $f'(c)$ cannot be positive.

Hence

$$f'(c)=0,$$

as required.

2°. **Theorem** (sufficient condition). *If in the neighbourhood 2δ of the point $x=c$:*

1) *the function $f(x)$ is continuous,*

2) *its derivative $f'(x)$ is positive to the left of c and negative to the right of c, then $x=c$ is a maximum point of the function.*

Proof. The given function is continuous at c; therefore (Sec. 69, 2°) the number $f(c)$ is the common limit for $f(c-\Delta x)$ and $f(c+\Delta x)$ when $\Delta x \to 0$ (as in the preceding theorem, here and henceforward $0<\Delta x<\delta$).

$$\lim_{-\Delta x\to 0} f(c-\Delta x)=f(c) \quad \text{and}$$
$$\lim_{+\Delta x\to 0} f(c+\Delta x)=f(c).$$

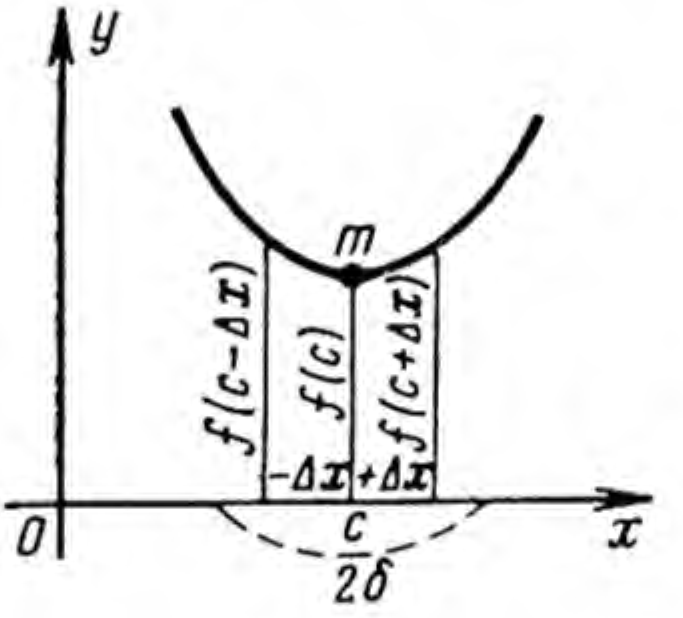

Fig. 115.

The given function $f(x)$ increases in the left semineighbourhood of c since its derivative is positive to the left, and decreases in the right semineighbourhood since here the derivative is negative (Fig. 114). Hence the values

$$f(c-\Delta x) \quad \text{and} \quad f(c+\Delta x)$$

increase as $\Delta x \to 0$. *

In other words, both $f(c-\Delta x)$ and $f(c+\Delta x)$ approach their limit $f(c)$ in such a way that for every value of $\Delta x \neq 0$

$$f(c-\Delta x)<f(c) \quad \text{and} \quad f(c+\Delta x)<f(c).$$

But then $f(c)$ is a maximum of the function $f(x)$ at the point $x=c$.

3°. In the same way it can be proved that *if in the neighbourhood 2δ of the point $x=c$:*

1) *the function $f(x)$ is continuous and* 2) *the derivative $f'(x)$ is negative to the left of the point $x=c$ and positive to the right, the value $x=c$ is a minimum point of the function* (Fig. 115).

4°. In both the maximum point and the minimum point the derivative equals zero (1°). But the converse is not true. *A function may have no maximum or minimum at a point where its derivative is zero.*

For example, the function $y=x^3$ has a derivative zero at the point $x=0$ (Sec. 99, 5°). However the point $x=0$ is neither a

* According to the definition of a decreasing function (Sec. 93), to a *lesser* value of the argument there corresponds a *greater* value of the function, i.e., when $x_1>x_2$, $f(x_1)<f(x_2)$.

maximum nor a minimum; the function $y=x^3$ increases for all values of x including $x=0$.

Hence it follows that the *function $f(x)$ has no maximum or minimum at $x=c$ if at $x=c$ its derivative equals zero, and has the same sign both to the left and to the right of the point $x=c$.*

5°. **Definition.** *The values of the argument x for which the derivative $f'(x)$ is zero are called stationary points.*

Tangents at such points are parallel to the x-axis.

In the neighbourhood of a maximum the tangent to the graph of the function makes an acute angle with the x-axis if the point of contact is to the left of the maximum, and an obtuse angle if the point of contact is to the right (Fig. 114). Conversely, in the case of a minimum the tangent forms an obtuse angle with the x-axis if the point of contact lies to the left of the minimum, and an acute angle if the point of contact lies to the right of the minimum (Fig. 115).

Sec. 103. Procedure for Finding Extremes

1°. To find the extreme of a function:

1) find the derivative of the given function;

2) equate the derivative to zero and solve the equation obtained; of the roots obtained, place the real roots in the order of increasing magnitude for the sake of convenience. If all the roots are imaginary the function has no extreme;

3) determine the sign of the derivative in each of the intervals bounded by stationary points;

4) if around a particular stationary point the derivative is positive to the left and negative to the right, the stationary point represents a maximum of the function; and if, on the other hand, the derivative is negative to the left and positive to the right of a stationary point, the latter point represents a minimum of the function. If the derivative has the same sign on either side of a stationary point, there is neither a maximum nor a minimum at this point;

5) in the expression for the function, substitute the argument (independent variable) by the value which corresponds to the maximum or minimum of the function. We thus obtain the maximum or minimum value of the function.

If the function has points of discontinuity, these points must be included in the stationary points that divide Ox into intervals over which the sign of the derivative is determined.

Sec. 104. Examples in Finding Extremes

1°. To find the maximum and minimum of the function

$$y=2+3x^2-x^3.$$

1) Find the derivative

$$y' = 6x - 3x^2.$$

2) Equate the derivative to zero and solve the equation obtained:

$$6x - 3x^2 = 0; \quad x(2-x) = 0; \quad x_1 = 0; \; x_2 = 2.$$

3) Points 0 and 2 divide the x-axis into three intervals:

$$1)\; -\infty < x < 0, \quad 2)\; 0 < x < 2 \quad \text{and} \quad 3)\; 2 < x < +\infty.$$

We factorise $6x - 3x^2$:

$$y' = -3x(x-2),$$

and determine the sign of the derivative in each of the intervals:

No. of interval	Characteristic of interval	Sign of $-3x$	Sign of $x-2$	Sign of y'
1	$-\infty < x < 0$	+	−	−
2	$0 < x < 2$	−	−	+
3	$2 < x < +\infty$	−	+	−

4) The derivative is negative in the interval lying to the left of the point $x=0$ and positive in the interval lying to the right of this point. Hence $x=0$ is a minimum point of the function.

The derivative is positive to the right and negative to the left of the point $x=2$. Hence this point is a maximum point of the function.

5) In the equation $y = 2 + 3x^2 - x^3$ substitute for x its values 0 and 2. Thus we obtain

$$y_{\min} = 2, \qquad y_{\max} = 6.$$

2°. Find the maximum and minimum points of the function

$$y = \frac{3x^2 + 5x + 25}{x+2}.$$

Solution. At $x = -2$ the function has no numerical value. The value -2 is a point of discontinuity. Noting this,

1) find the derivative:

$$y' = \frac{3(x^2 + 4x - 5)}{(x+2)^2};$$

2) the derivative must be equated to zero. But a fraction is zero when its numerator is zero. And the numerator is zero when

$$x^2 + 4x - 5 = 0.$$

Solving this equation we obtain

$$x_1 = -5; \quad x_2 = +1\,^*.$$

The point of discontinuity $x = -2$ is included in the points that divide the x-axis into intervals over which the sign of the derivative is determined.

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* It may happen that the numerator is a constant. This case is examined in Sec. 107.

3) The sign of the derivative at any point x, except at points of discontinuity, is determined by the sign of x^2+4x-5, since the coefficient 3 in the numerator of the derivative and the denominator of the derivative $(x+2)^2$ are both positive. For convenience factorise the expression

$$x^2+4x-5=(x+5)(x-1).$$

No. of interval	Characteristic of interval	Sign of $x+5$	Sign of $x-1$	Sign of y'
1	$-\infty<x<-5$	$-$	$-$	$+$
2	$-5<x<-2$	$+$	$-$	$-$
3	$-2<x<+1$	$+$	$-$	$-$
	$+1<x<+\infty$	$+$	$+$	$+$

4) The function has a maximum at $x=-5$ and a minimum at $x=1$.

3°. Test the following function for maximum and minimum:

$$y=\cos x\cdot\sin^3 x.$$

Solution.

$$1)\ y'=-\sin x\cdot\sin^3 x+3\sin^2 x\cdot\cos x\cdot\cos x$$

or

$$y'=3\sin^4 x\left(\cot^2 x-\frac{1}{3}\right)$$

or

$$y'=3\sin^4 x\left(\cot x+\frac{1}{\sqrt{3}}\right)\left(\cot x-\frac{1}{\sqrt{3}}\right). \tag{1}$$

2) Equating each factor separately to zero we obtain

$$\sin^4 x=0;\qquad \sin x=0;\qquad x=0;$$

$$\cot x+\frac{1}{\sqrt{3}}=0;\qquad \cot x=-\frac{1}{\sqrt{3}};\qquad x=-\frac{\pi}{3};$$

$$\cot x-\frac{1}{\sqrt{3}}=0;\qquad \cot x=\frac{1}{\sqrt{3}};\qquad x=\frac{\pi}{3}.$$

It may be noted that in solving trigonometric equations only the main values of the arguments are taken into account.

No. of interval	Characteristic of interval	Sign of $\cot x+\frac{1}{\sqrt{3}}$	Sign of $\cot x-\frac{1}{\sqrt{3}}$	Sign of y'
1	$-\frac{\pi}{2}<x<-\frac{\pi}{3}$	$+$	$-$	$-$
2	$-\frac{\pi}{3}<x<0$	$-$	$-$	$+$
3	$0<x<\frac{\pi}{3}$	$+$	$+$	$+$
4	$\frac{\pi}{3}<x<\frac{\pi}{2}$	$+$	$-$	$-$

Hence the function has a minimum at $x=-\frac{\pi}{3}$ and a maximum at $x=\frac{\pi}{3}$. At $x=0$ there is neither a maximum nor a minimum.

Sec. 105. Second Derivative Test for Extreme Values

1°. **Lemma.** *If at $x=c$ the derivative is positive (or negative), then in a sufficiently small neighbourhood of the point $x=c$ an increment in the function and an increment in the argument at $x=c$ have identical (or opposite) signs.*

Proof by reductio ad absurdum. For the sake of definiteness let $f'(c)>0$, i.e.,

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}>0.$$

Assume that Δy and Δx have opposite signs when $\Delta x\to 0$. Then $\frac{\Delta y}{\Delta x}$ is negative and its limit (Sec. 55) is

$$f'(c)\leqslant 0,$$

which contradicts the initial conditions of the problem.

The other part of the lemma is proved in the same way.

2°. **Theorem.** *If at $x=c$ the first derivative of the function $f(x)$ equals zero, $f'(c)=0$, and the second derivative is positive, $f''(c)>0$, then the function $f(x)$ has a minimum at $x=c$;*

if the second derivative is negative, $f''(c)<0$, then the function $f(x)$ has a minimum at $x=c$.

Proof. The second derivative has the same relation to the first derivative as the first derivative has to the function, i.e.,

$$f''(c)=\lim_{\Delta x\to 0}\frac{f'(c+\Delta x)-f'(c)}{\Delta x}.$$

In accordance with the lemma, if the derivative (in this case, the second derivative) is positive at $x=c$, then in a sufficiently small neighbourhood 2δ of the point c an increment in the function (in this case, the first derivative) has the same sign as the increment in the argument (independent variable). To the left of the point c an increment in the independent variable is negative; hence the increment in the function is also negative, i.e.,

$$f'(c-\Delta x)-f'(c)<0,\quad (0<\Delta x<\delta).$$

Whence

$$f'(c-\Delta x)<f'(c)=0. \tag{1}$$

To the right of c an increment in the independent variable is positive, i.e.,

$$f'(c+\Delta x)-f'(c)>0.$$

Whence

$$f'(c+\Delta x) > f'(c) = 0. \qquad (2)$$

We thus have that the first derivative of the function $f(x)$ on the left of the point c is negative (1) and on the right of the point, positive (2). This proves that at $x=c$ the function $f(x)$ has a minimum.

The theorem is likewise proved for the case when $f''(c) < 0$.

3°. The theorem that has just been proved is a second method for finding an extreme. It differs from the first one (Sec. 103) in that the third and fourth operations of the first method are replaced by: a) finding the second derivative and b) determining its sign at the stationary point. The result of the investigation may be stated thus:

If the sign of $f''(c)$ is	then at $x=c$, $f(x)$ has a
plus	minimum
minus	maximum

If $f''(c)=0$, the test of the function for maximum and minimum must be conducted by the first method.

4°. **Example 1.** Test the function $y=5-x^2-x^3-\frac{1}{4}x^4$ by the second method for maximum and minimum.

Solution. 1) Find the first derivative:

$$y' = -2x-3x^2-x^3.$$

2) Equate the first derivative to zero and solve the equation:

$$-2x-3x^2-x^3=0, \quad \text{or} \quad x(x^2+3x+2)=0,$$

whence $x=0$ or $x^2+3x+2=0$.

Solving the quadratic equation $x^2+3x+2=0$, we get

$$x=\frac{-3\pm 1}{2}.$$

Three stationary points are obtained: $x_1=-2$, $x_2=-1$, and $x_3=0$.

3) Find the second derivative:

$$y''=-2-6x-3x^2.$$

4) Determine the sign of the second derivative replacing x, successively, by its values in the first, second and third stationary points:

$$\text{when } x=-2 \quad y''=-2-6\cdot(-2)-3\cdot(-2)^2=-2,$$

$$\text{when } x=-1 \quad y''=-2-6\cdot(-1)-3\cdot(-1)^2=+1,$$

$$\text{when } x=0 \quad y''=-2.$$

Hence the given function has a minimum at $x=-1$ and a maximum at $x=-2$ and at $x=0$.

Example 2. Test for maximum and minimum the function

$$y=x^4.$$

Solution:

1) $y'=4x^3$;

2) $4x^3=0$; $x=0$;

3) $y''=12x^2$;

4) when $x=0$, $y''=0$.

Since the second derivative is found to be zero, the first method of testing is employed. When $x<0$, $y'=4x^3<0$, and when $x>0$, $y'=$ $=4x^3>0$. Hence the function $y=x^4$ has a minimum at the point $x=0$.

5°. The second method for finding the maximum and minimum should be employed when the second derivative is easily obtainable. If differentiation involves complex transformations and does not simplify the expression of the first derivative, the first method may lead to quicker results.

Sec. 106. Extreme Value Problems

1°. The difference between two numbers is a. What are they if their product is to be a minimum?

Solution. Let the smaller number be x. Then the larger one will be $x+a$. Their product is $x\ (x+a)$ and is a function of x. We denote it by y:

$$y=x^2+ax.$$

Find the value of x for which y is a minimum:

1) $y'=2x+a$; 2) $2x+a=0$; $x=-\frac{a}{2}$; 3) $y''=2$; 4) the function has a minimum at $x=-\frac{a}{2}$.

Thus, the product of two numbers differing by a is least if one of them is $-\frac{a}{2}$ and the other $+\frac{a}{2}$.

2°. The strength of a girder of rectangular cross-section is proportional to the product of its width by the square of its height. How is one to cut a girder of maximum strength out of a beam of d cm thickness?

Solution. Let the factor of proportionality be k (dependent on the material of the beam) and the strength y. Then by the statement of the problem

$$y=k\cdot b\cdot h^2$$

where (Fig. 116) b is the base and h is the altitude of the rectangle. The equation contains two independent variables, b and h; let us express h in terms of b. In triangle ABC, $AB=d$, $AC=b$ and $BC=h$. Hence

$$h^2=d^2-b^2.$$

Therefore,

$$y=k\cdot b\,(d^2-b^2), \quad \text{or} \quad y=d^2kb-kb^3.$$

Find the maximum of this function by the second method:

1) differentiating with respect to b, we get

$$\frac{dy}{db}=d^2k-3kb^2;$$

2) $d^2k-3kb^2=0; \quad b=\frac{d}{\sqrt{3}};$

3) $\frac{d^2y}{db^2}=-6kb;$

4) when $b=\frac{d}{\sqrt{3}}$ $y''=-\frac{6kd}{\sqrt{3}}<0$ (since $k>0$ and $d>0$),

the function has a maximum.

Thus the strength of the girder is a maximum if the width $b=\frac{d}{\sqrt{3}}$.

Let us determine the altitude:

$$h^2=d^2-b^2=d^2-\frac{d^2}{3}=\frac{2d^2}{3}.$$

Whence

$$h=\frac{d}{\sqrt{3}}\cdot\sqrt{2}.$$

Since $\frac{d}{\sqrt{3}}=b, \quad h=b\sqrt{2}.$

We have obtained a result of great practical importance: the strength of a girder of rectangular cross-section is greatest when its height is equal

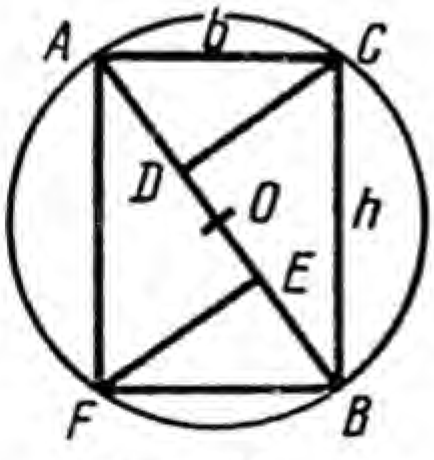

Fig. 116.

Fig. 117.

to the base multiplied by $\sqrt{2}$. This relation does not depend on the quality of the material as it is independent of k. Since $\sqrt{2}\approx 1.4=\frac{7}{5}$, the sides of the rectangle are approximately in the proportion $7:5$.

The following is a method for constructing a rectangle of maximum strength.

Divide the diameter AB (Fig. 116) by points D and E into three equal parts. From points D and E erect perpendiculars DC and EF on AB. The perpendiculars intersect the circle at C and F. Join C and F with A and B respectively. $ACBF$ is the required rectangle of maximum strength.

Indeed, the leg AC is a mean proportional between the hypotenuse AB and segment AD:

$$\frac{AB}{AC}=\frac{AC}{AD}; \qquad AC=\sqrt{AB\cdot AD}=\sqrt{d\cdot\frac{d}{3}}=\frac{d}{\sqrt{3}}.$$

Similarly

$$\frac{AB}{BC}=\frac{BC}{BD}; \qquad BC=\sqrt{AB\cdot BD}=\sqrt{d\cdot\frac{2}{3}d}=\frac{d}{\sqrt{3}}\cdot\sqrt{2}.$$

Hence, $BC=AC\cdot\sqrt{2}$.

3°. It is required to make a cylindrical vessel of aluminium (without a cover) of a given column v with the least expenditure of metal. What dimensions should it have?

Solution. The quantity of metal required depends on the surface of the vessel. Hence the problem reduces to finding the smallest surface that will enclose the given volume v. Let radius of the base of the cylinder be r and its height h (Fig. 117). We have two independent variables: r and h. Express h in terms of r.

The volume of a cylinder is $v=\pi r^2h$, whence $h=\frac{v}{\pi r^2}$. Let the surface of the cylinder be y. It consists of the area of the base, πr^2, and the area of the side,

$$2\pi rh=2\pi r\cdot\frac{v}{\pi r^2}=\frac{2v}{r}.$$

Hence $y=\pi r^2+\frac{2v}{r}$.

Let us find the minimum of this function noting that π and v are constants and r is the independent variable.

1) $\frac{dy}{dr}=2\pi r-\frac{2v}{r^2}=\frac{2(\pi r^3-v)}{r^2}$.

2) Equating the numerator of the derivative to zero and solving the equation, we get

$$r=\sqrt[3]{\frac{v}{\pi}}.$$

3) Take the derivative of $\frac{dy}{dr}=2\pi r-\frac{2v}{r^2}$:

$$\frac{d^2y}{dr^2}=2\pi+\frac{4v}{r^3}.$$

4) When $r=\sqrt[3]{\frac{v}{\pi}}$, $\quad \frac{d^2y}{dr^2}=2\pi+\frac{4v}{\frac{v}{\pi}}=6\pi>0$.

Hence, when $r=\sqrt[3]{\frac{v}{\pi}}$ the function has a minimum.

Let us determine h so as to establish the relation between r and h.

$$h=\frac{v}{\pi r^2}=\frac{v}{\pi}\cdot\frac{1}{r^2}=\frac{v}{\pi}\cdot\frac{1}{\sqrt[3]{\frac{v^2}{\pi^2}}}=\sqrt[3]{\frac{v}{\pi}},$$

i.e., $r=h$.

Thus, to make a cylindrical vessel (without a cover) with a minimum expenditure of metal, the height must be equal to the radius of its base.

Sec. 107. Maximum and Minimum of a Function at Points Where the Derivative Has No Value

1°. The function $y=|x|$ (Fig. 118) at $x=0$ is continuous and has a minimum. However it has no derivative at this point (Sec. 78). For all other points on the left or right of $x=0$ there exists a derivative, which we shall find by the general method (Sec. 75) as $\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$.

For the point x_1 $\Delta x>0$, $\Delta y<0$ and

$\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=-1$. For the point x_2, $\Delta x>0$, $\Delta y>0$ and

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=+1.$$

Thus, the sufficient condition for the existence of a minimum is satisfied for the function $y=|x|$ at the point $x=0$.

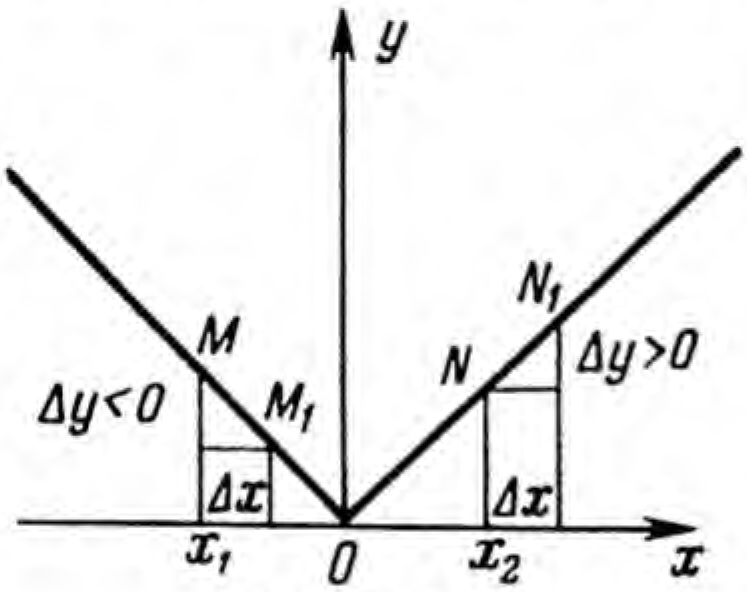

Fig. 118.

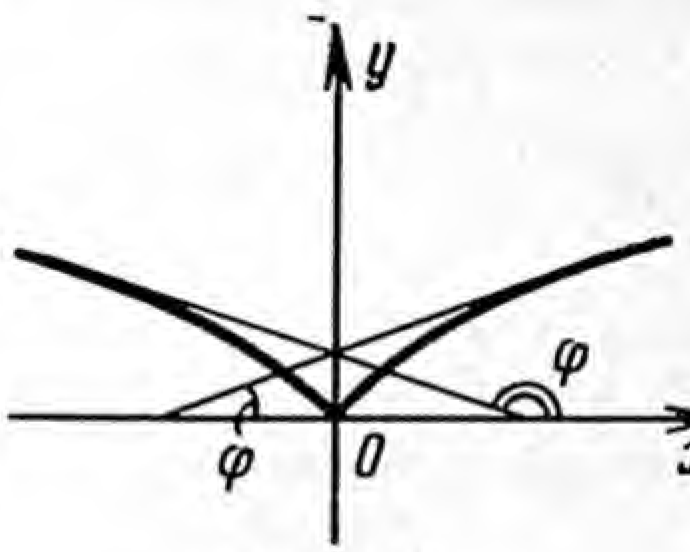

Fig. 119.

Hence a function can have an extreme at a point where the derivative of the function does not exist, yet the sufficient condition for the existence of an extreme is fulfilled. (Sec. 102, 2° and 3°.)

2°. The function $y=f(x)=\sqrt[3]{x^2}$ (Fig. 119) is continuous and has a minimum at the point $x=0$. However its derivative

$$f'(x)=\frac{2}{3\sqrt[3]{x}}$$

does not become zero at any value of x. On the left of $x=0$ $(x<0)$, the derivative is negative, on the right $(x>0)$ it is positive. Thus the sufficient condition for the existence of a minimum is fulfilled.

At the point $x=0$ the derivative equals infinity. Indeed, if $x<0$, $f'(x)\to-\infty$ as $x\to 0$; but if $x>0$, $f'(x)\to+\infty$ as $x\to 0$. The tangent to the graph of the function $y=\sqrt[3]{x^2}$ at $x=0$ is represented by two half-lines perpendicular to the axis Ox and bounded by the point O. This point is called a cusp.

If at the point under investigation the derivative is infinite, its reciprocal is

$$\frac{1}{f'(x)}=0.$$

Hence, to find all possible extremes of a function, including cusps, it is necessary to examine also the real roots of the equation $\frac{1}{f'(x)}=0$.

Sec. 108. The Direction of Concavity of a Curve

Let two points M_1 and M_2 have the same abscissa. If then the ordinate of M_1 is *larger* (smaller) than the ordinate of M_2 then it is said that M_1 lies *above* (below) M_2. It is also said that the line $y=f(x)$ lies *above* (below) the line $y=\varphi(x)$ in the interval $a<x<b$ if every point in this interval on the first line lies *above* (below) its corresponding point on the second line, i.e., if

$$f(x)>\varphi(x) \text{ [or } f(x)<\varphi(x)].$$

Definition. *A curve representing the graph of a differentiable function $y=f(x)$ is said to be concave upwards (downwards) over*

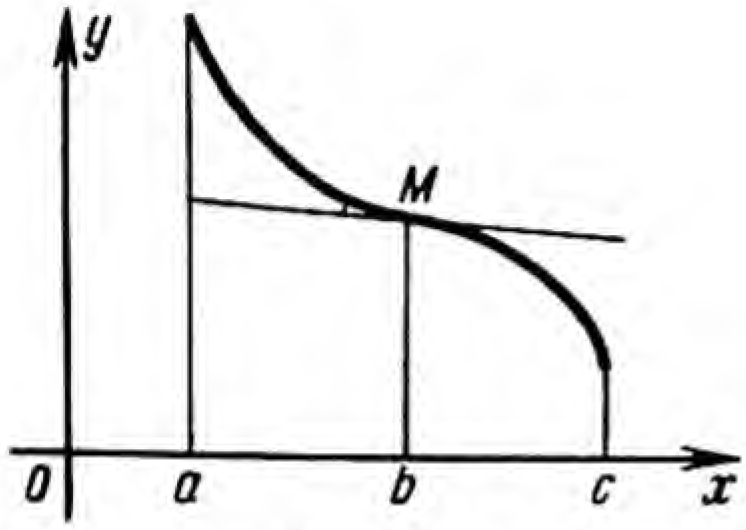

Fig. 120.

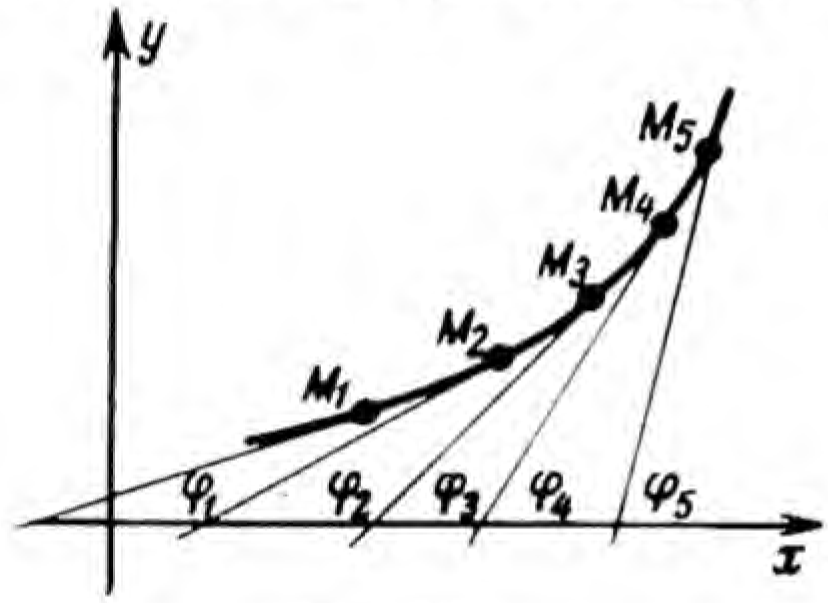

Fig. 121.

the interval $a<x<b$ if at every point of this interval the curve lies above (below) the tangent to the curve at that point.

The curve in Fig. 120 is concave upwards in the interval $a<$ $<x<b$ and concave downwards in the interval $b<x<c$.

2°. In more advanced courses of analysis it is shown that *a curve $y=f(x)$ is concave upwards (downwards) in the interval $a<x<b$ if its derivative, $f'(x)$, is an increasing (decreasing) function in this interval.*

To make this theorem clearer, mark, arbitrarily, a number of points on the axis Ox (Fig. 121) and draw a straight line through each of these points in such a manner that the slopes of the lines increase with the abscissas of the points. Then, taking these straight lines as tangents to a certain curve [$\tan\varphi=f'(x)$], construct this curve. It is evident that the curve can lie only above each of these tangents.

3°. **Sufficient condition for concavity upwards (downwards).** *If in the interval $a<x<b$ the second derivative $f''(x)$ is positive (negative), with the exception of individual points at which it is zero, then the curve of $y=f(x)$ is concave upwards (downwards) in this interval.*

Indeed, if in the interval $a<x<b$ the second derivative $f''(x)$ is, for example, positive (with the exception of individual

points where it is zero), then the first derivative $f'(x)$ is an increasing function (Sec. 99, 3°) and the curve $y=f(x)$ must be—from what has been just now demonstrated—concave upwards.

If $f''(x)=0$ not at some isolated (individual) points but over a certain interval, then, in this interval, $f'(x)$ is a constant function and $f(x)$ is a linear function whose graph is a straight line. And a straight line, of course, has no concavity.

Sec. 109. Points of Inflection

1°. **Definition.** *If in a certain neighbourhood of the point $x=c$ the curve—the graph of a differentiable function $y=f(x)$—has concavities in opposite directions to the left and right of the point $x=c$ the value of $x=c$ is called a point of inflection.*

The point M of a curve (Fig. 122) with abscissa $x=c$ is also called a point of inflection. M separates the arc of the curve concave upwards from the arc that is concave downwards. A point of inflection can only occur when the curve has a tangent. In the neighbourhood of a point of inflection the curve lies on either side of the tangent: above and below it. It should be noted that the curve also lies on both sides of the normal. But a point like P in Fig. 122, where the curve does not have one and only one tangent, is not a point of inflection.

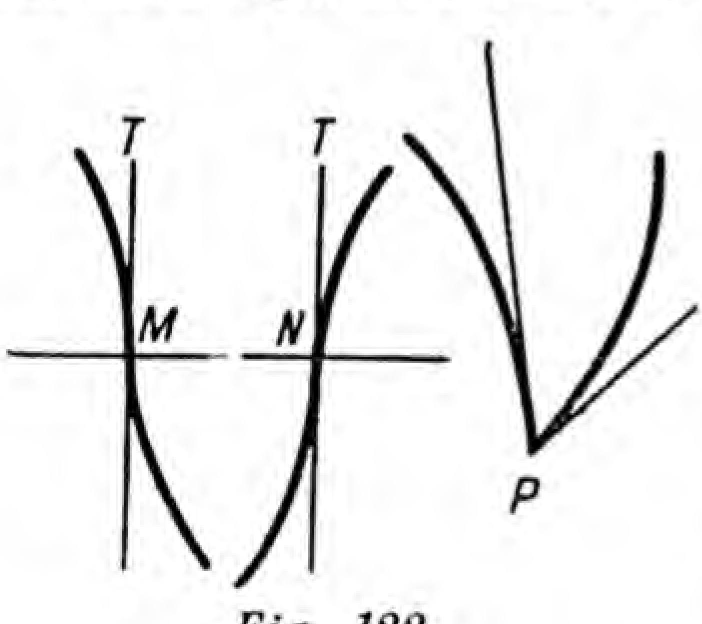

Fig. 122.

2°. Since the concavity of the curve $y=f(x)$ is of different direction on the left and right of the point of inflection, $x=c$, the second derivative $f''(x)$ is either zero or has opposite signs on the left and right of the point. Assuming the second derivative to be continuous in the neighbourhood of the point $x=c$, we conclude that its value at the point of inflection is zero, i.e.,

$$f''(c)=0.$$

3°. We thus have a **rule for finding points of inflection:**

1) *find the second derivative of the given function;*

2) *equate the second derivative to zero, solve the equation*, and write the real roots of this equation in the order of increasing magnitude;*

3) *determine the sign of the second derivative in each of the intervals bounded by the roots obtained;*

---

* Or find the values of x for which the second derivative has no numerical values.

4) *if then in two intervals bounded by the point under study the second derivative has opposite signs, there is a point of inflection; if the signs are the same, there is no point of inflection.*

4°. **Examples.**

1) Find the points of inflection and determine the intervals over which the curve of the function $y=\ln x$ is concave upwards and downwards.

Solution. Find the second derivative:

$$y'=\frac{1}{x}; \quad y''=-\frac{1}{x^2}.$$

At all values of $x=(0<x<+\infty)$, y'' is negative. Thus the logarithmic curve has no points of inflection and is concave downwards.

2) Investigate similarly the function $y=\sin x$.

Solution. Write the second derivative:

$$y'=\cos x; \quad y''=-\sin x.$$

Assuming $-\sin x=0$, we find that $x=k\pi$, where k is any integer. If $0<x<\pi$, $\sin x$ is positive and y'' is negative. If $\pi<x<2\pi$, $\sin x$ is negative and y'' is positive, etc. Thus a sinusoidal curve has points of inflection at $0, \pi, 2\pi, \ldots$

In the first interval $0<x<\pi$ its concavity is downwards, in the second interval $\pi<x<2\pi$ the concavity is upwards, etc.

Sec. 110. Constructing Graphs of Functions

1°. The graph of a function is plotted on the basis of an investigation which includes:

1) determining the domain of the function;

2) determining the points of discontinuity and the limits of the function to the left and right of these points;

3) finding the points of maximum and minimum;

4) defining the intervals over which the function increases or decreases;

5) finding the points of inflection; and

6) defining the intervals over which the curve is concave upwards and downwards.

2°. Construct the graph of the function $y=x^3-3x^2+4$.

Solution. Investigate the function.

1. The values of y are real numbers for all values of x, i.e., the domain is $-\infty<x<+\infty$.

2. There are no points of discontinuity, since a polynomial with constant coefficients is a continuous function.

3. We find the extreme:

$$y'=3x^2-6x=3x\cdot(x-2).$$

There are two stationary points: $x=0$ and $x=2$.

The function has a maximum at $x=0$ and a minimum at $x=2$; $y_{max}=4$, $y_{min}=0$.

Characteristic of interval	Sign of $3x$	Sign of $x-2$	Sign of y'
$-\infty < x < 0$	$-$	$-$	$+$
$0 < x < 2$	$+$	$-$	$-$
$2 < x < +\infty$	$+$	$+$	$+$

4. The function increases in the intervals: $-\infty < x < 0$ and $2 < x < +\infty$ and decreases in the interval $0 < x < 2$.

5. Find the points of inflection. $y'' = 6x - 6$ and becomes zero when $x = 1$. When $x < 1$, $y'' < 0$; when $x > 1$, $y'' > 0$. Thus the second derivative has opposite signs to the left and right of the point $x = 1$, which proves the latter to be a point of inflection.

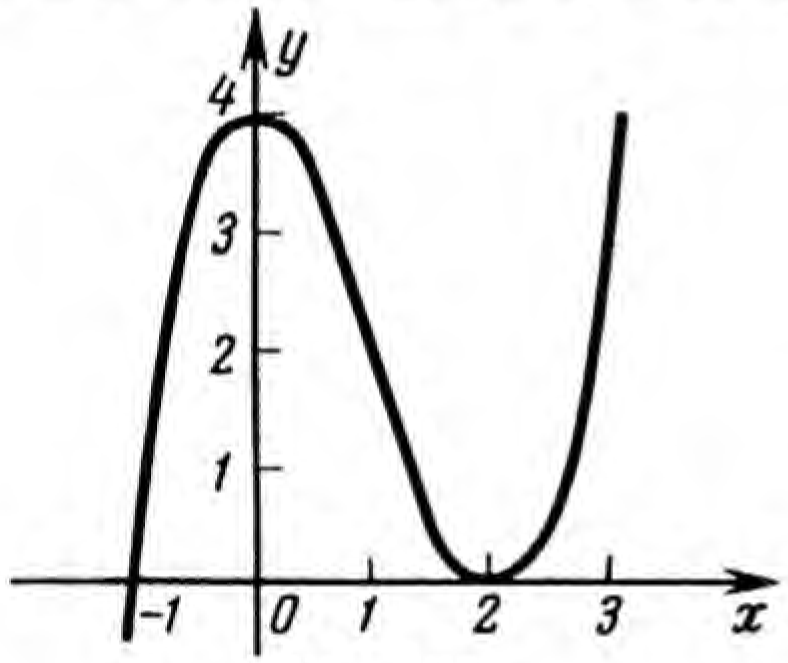

Fig. 123.

6. The curve is concave downwards in the interval $-\infty < x < +1$ and upwards in the interval $+1 < < x < +\infty$.

In a table of x- and y-values we write down the coordinates of the points of maximum, minimum and inflection so far found and give the values intermediate between them:

A graph of the function is given in Fig. 123.

If $x=$	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
then $y=$	$-6\frac{1}{8}$	0	$3\frac{1}{8}$	4	$3\frac{3}{8}$	2	$\frac{5}{8}$	0	$\frac{7}{8}$	4

Sec. 111. Mechanical Interpretation of the Second Derivative

Assume that a point moves in a straight line and the distance travelled is a function of the time: $s = f(t)$. The velocity v (Sec. 75) at time t is the derivative of the path with respect to time:

$$v = \frac{ds}{dt}.$$

The rate of change of velocity at time t is the acceleration a,

$$a = (v)' = \left(\frac{ds}{dt}\right)' = \frac{d^2s}{dt^2}.$$

The second derivative of the path with respect to time represents the acceleration of rectilinear motion at the given instant.

Example. $s=(t^3-2)$ metres is the law of rectilinear motion of a point. Determine the acceleration at $t=10$ sec.

Solution. The acceleration is $a=\frac{d^2s}{dt^2}$.

Differentiating the function $s=t^3-2$,

$$\text{we get } \frac{d^2s}{dt^2}=6t.$$

Hence,

$$a=6t=6\cdot 10=60; \quad a=60 \text{ m/sec}^2.$$

2°. If the motion is not uniform, the force F (producing the motion) is a variable quantity which is a function of time: $F=f(t)$, i.e., F has a certain definite value for any given instant of time t.

By Newton's law, the acting force F at any instant of time t is equal to the product of the mass m, by the acceleration, a, i.e.,

$$F=ma \quad \text{or} \quad f(t)=ma.$$

But in rectilinear motion $a=\frac{d^2s}{dt^2}$, therefore $f(t)=m\frac{d^2s}{dt^2}$.

Knowing the equation of rectilinear motion, it is possible to find the value of the acting force at any instant by the process of differentiation.

Example. Find the force which will cause a material point to oscillate rectilinearly under the law

$$s=A\cdot\sin(\omega t+\omega_0).$$

Solution. $f(t)=m\cdot\frac{d^2s}{dt^2}$,
from which we find the second derivative:

$$s=A\cdot\sin(\omega t+\omega_0),$$

$$\frac{ds}{dt}=A\cdot\cos(\omega t+\omega_0)\cdot\omega,$$

$$\frac{d^2s}{dt^2}=-A\cdot\sin(\omega t+\omega_0)\cdot\omega^2=-s\cdot\omega^2=-\omega^2 s; \quad f(t)=-m\omega^2 s.$$

Thus the oscillations under consideration occur due to a force proportional to the displacement s and acting in the opposite direction.

CHAPTER IX

DIFFERENTIAL

Sec. 112. Comparing Infinitesimals

1°. Let us write down a ratio of infinitesimals (that approach zero according to different laws) in such a manner that to any given instant in the approach to zero of one of the infinitesimals there corresponds some definite value of each of the infinitesimals under consideration. For example, let the corresponding values and relations at various times be such that:

when the value of	$\alpha = 10;\ 1;\ 0.1;\ 0.01$	etc.
the value of	$\beta = 1000;\ 1;\ 0.001;\ 0.000001$	etc.
and the ratio	$\frac{\beta}{\alpha} = 100;\ \ 1;\ 0.01;\ \ 0.0001$	etc.

i.e., the value of the ratio of infinitesimals in this example does not remain the same as the quantities approach zero. Thus the ratio of infinitesimals is a variable quantity that may have either a finite limit (zero, as in the above example, or some quantity other than zero) or a limit at infinity, or no limit at all.

2°. **Definitions.** 1) *An infinitely small quantity* β *is said to have a higher order of smallness than another such quantity* α *if the limit of the ratio* $\frac{\beta}{\alpha}$ *is zero, i.e., if*

$$\lim \frac{\beta}{\alpha} = 0;$$

2) *the infinitesimal* β *is said to be of a lower order of smallness than* α *if*

$$\lim \frac{\beta}{\alpha} = \infty;$$

3) β *and* α *are called infinitesimals of the same order if the limit of their ratio is the number* k, *some finite quantity other than zero, i.e., if* $\lim \frac{\beta}{\alpha} = k$, *where* $k \neq 0$ *and* $k \neq \infty$;

4) β *and* α *are called noncomparable infinitesimals if their ratio has no limit.*

3°. **Examples.** 1. In the above-cited example $\lim \frac{\beta}{\alpha}=0$, and β is of a higher order than α; $\lim\frac{\alpha}{\beta}=\infty$, α is of a lower order than β.

2. $\alpha=1-x$ and $\beta=1-x^2$ are infinitesimals if $x\to 1$. The ratio $\frac{\beta}{\alpha}=\frac{1-x^2}{1-x}=1+x$.

$$\lim\frac{\beta}{\alpha}=\lim_{x\to 1}(1+x)=2.$$

Thus, $1-x$ and $1-x^2$ are infinitesimals of the same order when $x\to 1$.

3. Let us compare $1-\cos x$ and x when $x\to 0$.

$$\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x}=\lim_{\frac{x}{2}\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\sin\frac{x}{2}\right)=$$

$$=\lim_{\frac{x}{2}\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\lim_{\frac{x}{2}\to 0}\sin\frac{x}{2}=1\cdot 0=0$$ (Sec. 44, 3° and Sec. 89, 3°),

i.e., $1-\cos x$, when $x\to 0$, is an infinitesimal of a higher order than x.

Sec. 113. Differential of a Function

1°. **Definition.** *The product of the derivative $f'(x)$ of a function $f(x)$ by an arbitrary increment Δx of the argument x is called the differential (dy) of the function $y=f(x)$, i.e.,*

$$\boxed{dy=f'(x)\cdot\Delta x}\tag{I}$$

2°. Thus, to obtain the value of the differential of a function it is necessary to know two numbers: the initial value of the argument x and its increment Δx.

Example. Calculate the differential of the function $y=x^2$ for a change in x from 3 to 3.1.

Solution. $dy=f'(x)\cdot\Delta x$. First write the expression for dy for arbitrary values of x and Δx.

$$f'(x)=(x^2)'=2x.$$

Hence $dy=2x\cdot\Delta x$.

The initial value of $x=3$, the increment $\Delta x=3.1-3=0.1$. Substituting these values into the expression for dy, we get

$$dy=2\cdot 3\cdot 0.1=0.6.$$

For a given value of the independent variable x, the differential of the function $f(x)$ is a linear function of the increment of the independent variable Δx.

3°. Let us consider the geometrical significance of the differential of a function. Fig. 124 shows the graph of a function $y=f(x)$ with a tangent at the point x. From ΔMPT it follows that

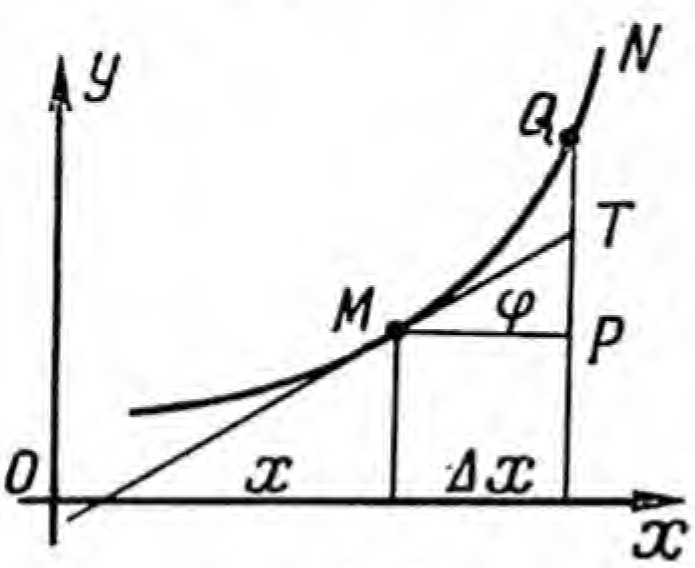

Fig. 124.

$$PT=MP\cdot\tan\varphi=\Delta x\cdot f'(x).$$

But by definition $f'(x)\cdot\Delta x=dy$. Therefore

$$PT=dy.$$

The differential of the function $f(x)$ for a given value of x is geometrically expressed by the increment in the ordinate of the tangent to the curve of the function $y=f(x)$ at the point x.

4°. The differential dy and the increment Δy of a function are, generally speaking, not equal. In Fig. 124 $dy=PT$ is less than $\Delta y=PQ$. Obviously, dy can also be greater than Δy. Such is the case, for example, if the rising curve MN is concave downwards.

5°. **Example.** For the function $y=x^2$, as x changes from 3 to 3.1, the increment $\Delta y=2x\cdot\Delta x+\Delta x^2=2\cdot 3\cdot 0.1+0.1^2=0.61$. The differential $dy=2x\cdot\Delta x=2\cdot 3\cdot 0.1=0.6$.

Taking dy as an approximate value of Δy, we find that the absolute error in the approximation is equal to the difference $\Delta y-dy=0.01$, and the relative error is the ratio

$$\frac{\Delta y-dy}{dy}=\frac{0.01}{0.60}=\frac{1}{60}=1.7\%.$$

6°. *The difference between the increment and the differential of a function, $\Delta y-dy$, is of a higher order than the increment of the argument, Δx.*

As has been observed in Sec. 49, the ratio $\frac{\Delta y}{\Delta x}$ differs from its limiting value $f'(x)$ by an infinitely small quantity α, and $\alpha\to 0$ as $\Delta x\to 0$,

$$\frac{\Delta y}{\Delta x}-f'(x)=\alpha.$$

Doing the subtraction on the left side of the equation, we get

$$\frac{\Delta y - f'(x)\cdot\Delta x}{\Delta x} = \alpha, \quad \text{or} \quad \frac{\Delta y - dy}{\Delta x} = \alpha,$$

$$\lim_{\Delta x \to 0} \frac{\Delta y - dy}{\Delta x} = \lim_{\Delta x \to 0} \alpha = 0.$$

7°. It follows from the foregoing that *the differential of a function is approximately equal to its increment, the relative error tending to zero along with the increment of the independent variable (argument).*

8°. Whence it follows that *the differential dy of a function $y = f(x)$ has two properties*:

1) *dy is proportional to Δx ($dy = k\Delta x$, where $k = y'$);*
2) *the ratio $\frac{\Delta y - dy}{\Delta x}$ tends to zero as Δx tends to zero.*

Conversely. *If a quantity z possesses two properties such that*

$$1)\ z = k\cdot\Delta x \quad \text{and} \quad 2)\ \lim_{\Delta x \to 0} \frac{\Delta y - z}{\Delta x} = 0,$$

then z is the differential of the function y.

Proof. Substituting into expression (2) the value of z in expression (1), we get

$$\lim_{\Delta x \to 0} \frac{\Delta y - k\cdot\Delta x}{\Delta x} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x} - k\right) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} - \lim k = y' - k = 0,$$

i.e.,

$$k = y',$$

and, consequently,

$$z = k\cdot\Delta x = y'\cdot\Delta x,$$

i.e., z is the differential of the function y.

Thus these two conditions fully determine a differential.

Sec. 114. The Differential of an Argument. The Derivative as a Ratio of Differentials

1°. **Definition.** *The increment Δx in the argument x is called the differential (dx) of the argument:*

$$\boxed{dx = \Delta x} \qquad \text{(II)}$$

Perhaps the notation is somewhat justified by the fact that the differential of the function $y = x$ and the increment of the argument are one and the same thing.

Indeed,

$$dy = (x)'\cdot\Delta x \quad \text{or} \quad dy = \Delta x.$$

But since

$$dy = dx, \quad dx = \Delta x,$$

i.e., the differential of the function $y = x$ and the increment of its argument are identical.

2°. Substituting $\Delta x = dx$ into formula (I), we obtain

$$\boxed{dy = f'(x) \cdot dx} \qquad \text{(III)}$$

i.e., *the differential of a function is the product of its derivative by the differential of the argument.*

3°. Formula (III) has a notable quality, viz., *formula $dy = f'(x)\,dx$ holds even when x is not an independent variable but is itself a function of another independent variable, say u.*

Indeed, if x is a function of u, $f(x)$ is a composite function of u; the dx is dependent on the increment Δu; and dy has to be evaluated from the formula

$$dy = f'_u(x) \cdot \Delta u,$$

but

$$f'_u(x) = f'_x(x) \cdot x'_u \text{ (Sec. 88).}$$

Therefore,

$$dy = f'(x) \cdot x'_u \cdot \Delta u.$$

But since by definition

$$x'_u \cdot \Delta u = dx,$$

it follows that

$$dy = f'(x) \cdot dx.$$

4°. **Example.** Find the differential of the function

$$y = \sqrt{e^{2x} - 1}.$$

Solution. By formula (III)

$$dy = y' \cdot dx$$

we find y':

$$y' = \frac{1}{2\sqrt{e^{2x}-1}} \cdot e^{2x} \cdot 2 = \frac{e^{2x}}{\sqrt{e^{2x}-1}}.$$

Hence

$$dy = \frac{e^{2x} \cdot dx}{\sqrt{e^{2x}-1}}.$$

5°. It follows from formula (III) that

$$\boxed{f'(x) = \frac{dy}{dx}}$$

i.e., *the derivative of a function is equal to the ratio of the differential of the function to the differential of the argument.* This is illustrated in Fig. 130 where

$$\frac{dy}{dx}=\frac{PT}{MP}=\tan\varphi=f'(x)$$

for an arbitrary value of $dx=MP$.

Sec. 115. Applying the Concept of Differential to Approximate Calculations

1°. The difference $\Delta y-dy$ is a higher order infinitesimal than Δx; hence, for a sufficiently small value of Δx we have

$$\boxed{\Delta y\approx dy=f'(x)\,\Delta x} \qquad \text{(IV)}$$

This means that *for small changes in the value of the argument (from its initial value x), the change in the value of the function $y=f(x)$ can be taken to be approximately proportional to the amount of change in the argument, the proportionality factor being equal to the value of the derivative $f'(x)$; and the curve $y=f(x)$, in such a case, can be approximately replaced by the tangent to it at the point x.*

Since $\Delta y=f(x+\Delta x)-f(x)$, by replacing Δy by its expression in formula (IV), we get

$$f(x+\Delta x)-f(x)\approx f'(x)\,\Delta x,$$

$$\boxed{f(x+\Delta x)\approx f(x)+f'(x)\cdot\Delta x} \qquad \text{(V)}$$

2°. Formulas (IV) and (V) are useful in many applications of the differential concept. Here we shall consider only applications in the field of approximate calculations.

In practice a measurement gives only an approximate value of a quantity. Let x be the approximate value of an argument obtained as a result of its measurement, and let $x+\Delta x$ be its true value. Then x determines the approximate value of the function $f(x)$, and $x+\Delta x$ gives the value of the function, $f(x+\Delta x)$.

The absolute difference between the true and approximate values is called the *absolute error*. The absolute error of the argument is equal to $|\Delta x|$. And the absolute error of the function is

$$|\Delta y|=|f(x+\Delta x)-f(x)|.$$

The absolute value of the ratio of absolute error to the value of a quantity is called the *relative error*.

The ralative error in determining the value of a function is equal to

$$\left|\frac{\Delta y}{y}\right|^{*}.$$

Determination of errors in measurements is one of the most important problems in engineering practice, particularly when the measurements are repeated many times, thus involving repeated errors due to a variety of conditions, such as, for example, geodetic measurements in surveying. Just as important is determining errors in calculation, in as much as the relative error appearing in the result of an operation (subtraction, multiplication, etc.) is different from the relative error in the initial quantities which are operated on.

To find the relative error of a function it is necessary first of all to find $\Delta y = f(x+\Delta x) - f(x)$. The right side of the equation $y = f(x)$ is frequently a complex mathematical expression; and so finding Δy demands great skill and involves intricate transformations. Yet an approximate value of Δy (the differential of the function dy) can be found from any form of the function without any special difficulty by the use of the formulas that have been studied. For this reason, the increment Δy is usually replaced by the differential, and the relative error δ is taken equal to $\left|\frac{dy}{y}\right|$, i.e.,

$$\delta = \frac{dy}{y}. \tag{VI}$$

3°. Examples. 1. Let us demonstrate how comparatively easy it is to find the tabular difference of decimal logarithms. The tabular difference Δy is the increment in the decimal logarithm

$$y = \log_{10} x$$

as x increases by 1 and is approximately equal to the linear increment dy (formula IV):

$$\Delta y \approx dy = (\log_{10} x)' \cdot dx = \frac{dx}{x} \cdot \frac{1}{\ln 10}.$$

Since

$$\frac{1}{\ln 10} = 0.43429 \text{ and } dx = \Delta x,$$

$$\Delta y \approx 0.43429 \cdot \frac{\Delta x}{x}.$$

Assuming $\Delta x = 1$ and $x = N$, we find that the tabular difference is

$$\Delta y \approx \frac{0.43429}{N}.$$

* y in this formula signifies the approximate va ue of the function, i.e., $f(x)$ taken with a defect.

2. Find the logarithm of a number N_1 which lies between two successive numbers N and $N+1$ given in the table.

In the formula for the increment of a logarithm

$$\Delta y \approx 0.43429 \cdot \frac{\Delta x}{x}$$

we assume $x=N$, $\Delta x = N_1 - N < 1$. Then the correction Δy to $\log N$ is given by the formula

$$\Delta y \approx \frac{0.43429}{N} \cdot (N_1 - N).$$

This increment of the logarithm appears in the columns of proportional differences.

The approximate value of $\log N_1$ (formula V) is

$$\log N_1 \approx \log N + \frac{0.43429}{N} \cdot (N_1 - N).$$

3. Let us determine the relative error occurring when a number is sought from its logarithm.

Assume that the logarithm in question of number x was calculated with an error Δy, which in turn will produce an error Δx in finding the number x from the logarithm. Then the relative error in x is

$$\left| \frac{\Delta x}{x} \right| .$$

From the formula

$$\Delta y \approx 0.43429 \cdot \frac{\Delta x}{x}$$

we get

$$\left| \frac{\Delta x}{x} \right| \approx \frac{|\Delta y|}{0.43429},$$

i.e., the relative error produced when a number is found from its logarithm is equal to

$$\frac{|\Delta y|}{0.43429}$$

and is independent of the value of the number, being dependent solely on the error with which the logarithm of number x was taken. If the logarithm of the number is accurate to five places of decimals,

$$|\Delta y| \leqslant \frac{1}{2} 0.00001,$$

then the maximum relative error is

$$\left| \frac{\Delta x}{x} \right| \approx \frac{1}{0.43429 \cdot 2 \cdot 100\,000} = \frac{1}{86858} .$$

If the logarithm is a four-digit number, then $\left|\frac{\Delta x}{x}\right| \approx \frac{1}{8686}$. The absolute error in finding the number x is

$$|\Delta x| \approx \frac{1}{86858} \cdot |x|.$$

Whence we conclude that the fifth figure of a number cannot be accurate if the number exceeds 86858 and is found in a five-place table; and similarly for the fourth figure if the number exceeds 8686 and is found in a four-place table. When looking for a number in a five-place table, the sixth figure can never be trusted, and similarly for the fifth figure in any number when using a four-place table. It is therefore pointless to look for, say, the sixth or seventh figure of a number when using a five-place table.

4. **Theorem.** *The relative error of a product does not exceed the sum of the relative errors of its factors.*

Proof. Let $y = u \cdot v$. Taking logarithms of the two sides and finding the differential, we get

$$\ln y = \ln u + \ln v;$$

$$\frac{dy}{y} = \frac{du}{u} + \frac{dv}{v}.$$

Whence

$$\left|\frac{dy}{y}\right| \leqslant \left|\frac{du}{u}\right| + \left|\frac{dv}{v}\right|.$$

Since $\left|\frac{dy}{y}\right|$ is the relative error of the product and $\left|\frac{du}{u}\right|$ and $\frac{dv}{v}$ are the relative errors of the factors, the theorem is proved.

5. **Theorem.** *The relative error of a quotient does not exceed the sum of the relative errors of the dividend and divisor.*

Proof. Let $y = \frac{u}{v}$.

Taking logarithms and the differential, we get

$$\ln y = \ln u - \ln v;$$

$$\frac{dy}{y} = \frac{du}{u} - \frac{dv}{v}.$$

Whence

$$\left|\frac{dy}{y}\right| \leqslant \left|\frac{du}{u}\right| + \left|\frac{dv}{v}\right|$$

as required.

If the maximum relative error is being sought, the sign $\leqslant$ can be replaced by the equality sign.

C. ELEMENTS OF INTEGRAL CALCULUS

CHAPTER X

INDEFINITE INTEGRAL

Sec. 116. Integration as the Inverse of Differentiation

1°. Differentiation consists in finding the derivative or differential of a given function. Integration is the inverse operation. *The purpose of integration is to find the functions of which the given function $f(x)$ is the derivative.*

Example. To find the function of which x^2 is the derivative.

Solution. Let the required function be $F(x)$. It is given that $F'(x)=x^2$; we surmise that $F(x)=\frac{x^3}{3}$ since $F'(x)=\frac{3x^2}{3}=x^2$.

This function, $\frac{x^3}{3}$, is called the antiderivative or the integral of x^2.

Let it be noted that if to the function $\frac{x^3}{3}$ any number 1, -2, etc., is added, the resultant functions $\frac{x^3}{3}+1$, $\frac{x^3}{3}-2$, etc., also serve as the answer to our problem, since the derivative of each of these functions is equal to x^2:

$$\left(\frac{x^3}{3}+1\right)'=x^2;\ \left(\frac{x^3}{3}-2\right)'=x^2.$$

Hence it follows that there is not one antiderivative of x^2 but an infinite number consisting of one and the same function $\frac{x^3}{3}$ to which some constant is added. Let us denote this arbitrary constant by c. Then the antiderivatives for x^2 are functions of the form

$$\frac{x^3}{3}+c.$$

2°. In specific problems the ambiguity of the answer is eliminated by some supplementary condition.

Example. To find the function whose derivative is x^2 and whose value is 5 when $x=3$.

Solution. The supplementary condition consists in the fact that the value of the antiderivative, which, as we have already noted, is $\frac{x^3}{3}+c$, is equal to 5 if $x=3$. Substituting 3 for x in $\frac{x^3}{3}+c$, we obtain

$$\frac{3^3}{3}+c=5, \text{ and therefore } c=-4.$$

Hence there is one required function:

$$\frac{x^3}{3}-4.$$

3°. In practice it is often required to find quantities from their given derivatives. Let us study a few examples.

Example 1. The velocity of a body at every instant of time t is equal to t^2 m/sec. Determine the distance travelled in t sec from the start if the body started from rest.

Solution. The velocity at time t is the derivative of the distance with respect to time, $\frac{ds}{dt}$; thus

$$\frac{ds}{dt}=t^2.$$

Whence it follows that

$$s=\frac{t^3}{3}+c.$$

To determine c utilise the initial condition that the body started from rest. Hence if $t=0$, $s=0=\frac{0}{3}+c$, from which it follows that $c=0$.

Thus the sought distance

$$s=\frac{t^3}{3}.$$

Example 2. The slope of a tangent at every point of a curve is equal to $2x$. Find the equation of the curve if it is given that the curve passes through the point (2, 7).

Solution. The slope of a tangent is the tangent of the angle which the tangent to the curve makes with the x-axis and is equal to the derivative of the function of the curve $y=F(x)$. Here

$$\frac{dy}{dx}=2x.$$

Guessing, we find that

$$y=x^2+c.$$

This equation describes an infinite number of parabolas $y = x^2$ (Fig. 125) spaced successively on the y-axis in such manner that the ordinate of the vertex of the parabola in each case is equal to c (if $x = 0$, $y = c$). In the figure, a tangent is drawn to each parabola at $x = 1$; the tangents are parallel to each other since under the conditions of the problem they have one and the same slope $k = 2x = 2 \cdot 1 = 2$.

The problem also states that the required curve passes through the point (2, 7). Therefore the coordinates (2, 7) satisfy the equation $y = x^2 + c$. Substituting 2 and 7 for the coordinates x and y in this equation, we get

$$7 = 2^2 + c; \quad c = 3.$$

Hence the equation of the required curve is

$$y = x^2 + 3.$$

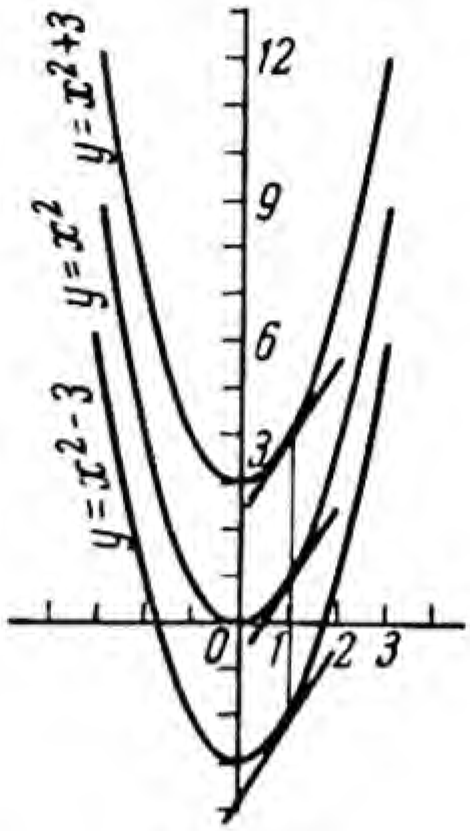

Fig. 125

Sec. 117. The Indefinite Integral as an Expression of the Aggregate of Antiderivatives of a Given Function

1°. **Definition.** *The antiderivative, or integral, of a given function is a function whose derivative is equal to the given function.*

Thus $F(x)$ is the antiderivative, or integral, of the function $f(x)$ if

$$F'(x) = f(x). \tag{1}$$

The question naturally arises: has every function an antiderivative?

In advanced courses of analysis it is shown that every function continuous on the interval $[a, b]$ has an antiderivative.

Henceforward we shall always assume that the given function $f(x)$ is continuous.

2°. **Theorem.** *A function differing from a differentiable function by an arbitrary number has the same derivative as the latter function.*

Indeed, if $\Phi(x) = F(x) + c$, where c is an arbitrary number and $F(x)$ is a differentiable function, then

$$\Phi'(x) = F'(x) + c' = F'(x)$$

(since $c' = 0$), as required.

3°. **Corollary.** If $F(x)$ is the antiderivative of the given function $f(x)$, then all functions which are obtained by the addition of

an arbitrary constant c to $F(x)$, i.e., all functions of the form $F(x)+c$, are also antiderivatives of the given function $f(x)$ since their derivatives are also equal to $f(x)$.

Thus, *for a given continuous function $f(x)$ there is an infinite number of antiderivatives, not one.*

4°. **Converse Theorem.** *The difference between any two antiderivatives having the same derivative is a constant.*

Proof. By the statement of the problem both functions $\Phi(x)$ and $F(x)$ have the same derivative $f(x)$:

$$\Phi'(x)=f(x) \text{ and } F'(x)=f(x).$$

$\Phi(x)$ and $F(x)$ are differentiable functions and therefore their difference, $\Phi(x)-F(x)$, is also a differentiable function (Sec. 86). Let us denote this difference by the letter y:

$$y=\Phi(x)-F(x).$$

Differentiating, we have

$$y'=[\Phi(x)-F(x)]'=\Phi'(x)-F'(x)=f(x)-f(x)=0.$$

Consequently (Sec. 99, 1°), the difference $\Phi(x)-F(x)$ is a constant. We denote it by the letter c:

$$\Phi(x)-F(x)=c.$$

5°. **Corollary.** *If for a given function $f(x)$, one antiderivative $F(x)$ is found, then any other one is obtained by adding a certain constant c; it has the form $F(x)+c$.*

For an arbitrary c, $F(x)+c$ is a general expression for the set of all antiderivatives of the given function $f(x)$.

6°. **Definition.** *The set of all functions whose derivative is equal to $f(x)$ is denoted by the symbol $\int f(x)\,dx$ and is called the indefinite integral of the function $f(x)$.*

The symbol $\int f(x)\,dx$ reads "indefinite integral of the function $f(x)\,dx$".

By definition,

$$\int f(x)\,dx=F(x)+c. \tag{2}$$

In equality (2) the symbol $\int$ is called the sign of integration, $f(x)$ is the integrand, $f(x)\cdot dx$ is the expression under the integral sign, $F(x)$ is called the functional part of the indefinite integral, and c is an arbitrary constant of the indefinite integral.

7°. *To find the indefinite integral of any function it is sufficient to find one of its antiderivatives and add to the latter an arbitrary constant c.*

Examples.

$$1)\ \int x^2\,dx=\frac{x^3}{3}+c; \qquad 2)\ \int 2x\,dx=x^2+c;$$

$$3)\ \int \cos x\,dx=\sin x+c,$$

since $(\sin x)'=\cos x$.

Sec. 118. Properties of an Indefinite Integral

1°. It follows from the equalities $F'(x)=f(x)$ and $\int f(x)\,dx=$ $=F(x)+c$ that

a) *the derivative of an indefinite integral is equal to the integrand*:

$$\left[\int f(x)\,dx\right]'=f(x);$$

b) *the differential of an indefinite integral is equal to the expression under the integral sign*:

$$d\int f(x)\,dx=f(x)\,dx;$$

c) *the indefinite integral of the differential of a function is equal to the sum of this function and an arbitrary constant.*

Indeed, let $F(x)$ be some *definite* antiderivative of the function $f(x)$, i.e., $f(x)=F'(x)$.

Then

$$f(x)\,dx=F'(x)\,dx=dF(x);$$

$$\int f(x)\,dx=\int dF(x).$$

But since

$$\int f(x)\,dx=F(x)+c,$$

it must follow that

$$\int dF(x)=F(x)+c.$$

The last two properties can be stated thus: the differential sign cancels the sign of indefinite integration, and the sign of indefinite integration cancels the differential sign but adds an arbitrary constant.

2°. *The indefinite integral of an algebraic sum of several functions is equal to the same algebraic sum of the indefinite integrals of the individual terms of the first sum*; for example,

$$\int (z-u+v)\,dx=\int z\,dx-\int u\,dx+\int v\,dx,$$

since

$$\left[\int (z-u+v)\,dx\right]'=z-u+v$$

and

$$\left[\int z\,dx-\int u\,dx+\int v\,dx\right]'=\left[\int z\,dx\right]'-\left[\int u\,dx\right]'+$$
$$+\left[\int v\,dx\right]'=z-u+v.$$

3°. *The constant factor of the integrand can be placed outside the sign of indefinite integration, i.e., if A is some constant, then*

$$\int A\cdot f(x)\,dx=A\cdot\int f(x)\,dx,$$

since

$$\left[\int A\cdot f(x)\,dx\right]'=A\cdot f(x)$$

and

$$\left[A\cdot\int f(x)\,dx\right]'=A\cdot\left[\int f(x)\,dx\right]'=A\cdot f(x).$$

Sec. 119. Integration by Formulas

1°. 1. Since $d(x+c)=dx$, therefore $\int dx=x+c$.

2. Since $d\left(\frac{x^{n+1}}{n+1}+c\right)=x^n\,dx$, $\int x^n\,dx=\frac{x^{n+1}}{n+1}+c$. The formula is true for $n\neq-1$; when $n=-1$, the expression $\frac{x^{n+1}}{n+1}$ loses numerical meaning because the denominator becomes zero, and one cannot divide by zero.

3. Since $d(\ln x+c)=\frac{1}{x}\cdot dx=\frac{dx}{x}$, $\int x^{-1}\,dx=\int\frac{dx}{x}=\ln x+c$.

4. Since $d(e^x+c)=e^x\,dx$, $\int e^x\,dx=e^x+c$.

5. Since $d\left(\frac{a^x}{\ln a}+c\right)=a^x\,dx$, $\int a^x\,dx=\frac{a^x}{\ln a}+c$.

2°. **Examples.** 1. Evaluate $\int x^5\,dx$.

Solution. By formula (2):

$$\int x^5\,dx=\frac{x^{5+1}}{5+1}+c=\frac{x^6}{6}+c;$$

2. $\int\sqrt[3]{x^2}\,dx=\int x^{\frac{2}{3}}\,dx=\frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1}+c=\frac{x^{\frac{5}{3}}}{\frac{5}{3}}+c=\frac{3}{5}x\sqrt[3]{x^2}+c.$

3. Evaluate $\int (x+1)(x-2)\,dx$.

Solution. Opening the brackets in the integrand, we get

$$\int (x^2-x-2)\,dx.$$

Then replacing the integral of the sum by the sum of the integrals, we have

$$\int x^2\,dx-\int x\,dx-\int 2dx.$$

The constant factor, 2, in the third integral is taken outside of the integral sign:

$$\int x^2\,dx-\int x\,dx-2\int dx.$$

Applying formulas (2) and (1), we have

$$\int (x+1)(x-2)\,dx=\int x^2\,dx-\int x\,dx-2\int dx=$$
$$=\frac{x^3}{3}-\frac{x^2}{2}-2x+c.$$

4. Evaluate $\int \frac{x^2-2x+3}{x^3}\,dx$.

Solution. Dividing each of the terms of the numerator by x^3, we get

$$\int \frac{x^2-2x+3}{x^3}\,dx=\int (x^{-1}-2x^{-2}+3x^{-3})\,dx=$$
$$=\int \frac{dx}{x}-2\int x^{-2}\,dx+3\int x^{-3}\,dx=\ln x-2\cdot\frac{x^{-1}}{-1}+$$
$$+3\cdot\frac{x^{-2}}{-2}+c=\ln x+\frac{2}{3}-\frac{3}{2x^2}+c.$$

3°. The method of integration which converts the given integral into a sum of integrals is called the *expansion method*. This method was used in examples 3 and 4 above.

Sec. 120. Integration by Substitution

To reduce a given integral to "tabular" form, i.e., as given in tables of integrals, use is sometimes made of the *method of substitution of a new variable.*

1°. **Theorem.** *Let $F(u)$ be an antiderivative for the function $f(u)$. Replacing the argument u by a function of the independent variable x having a continuous derivative,*

$$u=\varphi(x),$$

we have

$$\int f(u)\,du = \int f[\varphi(x)]\,\varphi'(x)\,dx. \tag{1}$$

Proof. Replacing u by the function $\varphi(x)$, we get the composite function

$$f(u) = f[\varphi(x)].$$

The integrand $f(u)\,du$ is the differential of the antiderivative:

$$f(u)\,du = dF(u) = F'(u)\,du.$$

But the formula of the differential holds even if u is the function of a different argument (Sec. 114, 3°),

$$f(u)\,du = F'(u)\,du = F'(u)\cdot\varphi'(x)\cdot dx,$$

since

$$du = d\varphi(x) = \varphi'(x)\,dx.$$

But it is given that

$$F'(u) = f(u) = f[\varphi(x)].$$

Therefore

$$f(u)\,du = f[\varphi(x)]\cdot\varphi'(x)\cdot dx.$$

Consequently,

$$\int f(u)\,du = \int f[\varphi(x)]\cdot\varphi'(x)\cdot dx.$$

2°. In actual integration, use has to be made of equality (1) written in the reverse order:

$$\int f[\varphi(x)]\cdot\varphi'(x)\,dx = \int f(u)\,du.$$

3°. Evaluate $\int (2x-3)^{\frac{1}{2}}\,dx$.

Solution. Since the integrand is a composite function, let us introduce a new variable and put

$$\varphi(x) = 2x - 3 = u,$$

$$\varphi'(x)\,dx = 2dx = du; \quad dx = \frac{1}{2}\,du.$$

Introducing these expressions into the integral, we get

$$\int (2x-3)^{\frac{1}{2}}\,dx = \int u^{\frac{1}{2}}\cdot\frac{1}{2}\,du = \frac{1}{2}\int u^{\frac{1}{2}}\,du = \frac{1}{2}\,\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c =$$

$$= \frac{1}{3}\,u^{\frac{3}{2}} + c = \frac{1}{3}\,(2x-3)^{\frac{3}{2}} + c.$$

4°. Evaluate $\int \frac{dx}{3x+5}$.

Solution. Introducing a new variable, we have

$$\varphi(x) = 3x + 5 = u;$$

$$\varphi'(x)\,dx = 3 \cdot dx = du; \quad dx = \frac{1}{3}\,du;$$

$$\int \frac{dx}{3x+5} = \int \frac{\frac{1}{3}\,du}{u} = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln u + c = \frac{1}{3}\ln(3x+5) + c =$$

$$= \ln \sqrt[3]{3x+5} + c.$$

5°. Evaluate $\int \frac{dx}{(2x-1)^2}$.

Solution. Let $2x - 1 = u$. Taking the differentials, we get $2dx = du$, $dx = \frac{1}{2}\,du$.

$$\int \frac{dx}{(2x-1)^2} = \int \frac{\frac{1}{2}\,du}{u^2} = \frac{1}{2}\int u^{-2}\,du = -\frac{1}{2u} + c = \frac{1}{2(1-2x)} + c.$$

6°. Evaluate $\int e^{3x}\,dx$.

Solution. To reduce the integral to formula (4) put $3x = u$. Then $dx = \frac{1}{3}\,du$:

$$\int e^{3x}\,dx = \int e^u \cdot \frac{1}{3}\,du = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + c = \frac{1}{3}e^{3x} + c.$$

7°. Evaluate $\int a^{nx}\,dx$.

Solution. To reduce the integral to formula (5), put $nx = u$. Then $n\,dx = du$, $dx = \frac{1}{n}\,du$.

$$\int a^{nx}\,dx = \int a^u \cdot \frac{1}{n}\,du = \frac{1}{n}\int a^u\,du = \frac{1}{n}\cdot\frac{a^u}{\ln a} + c = \frac{a^{nx}}{n\ln a} + c.$$

8°. Evaluate $\int \frac{2x\,dx}{x^2+1}$.

Solution. The integrand $\frac{2x}{x^2+1}$ is a fraction. Putting the denominator $x^2 + 1 = u$ and taking differentials of both sides, we get the numerator $2x\,dx = du$.

$$\int \frac{2x\,dx}{x^2+1} = \int \frac{du}{u} = \ln u + c = \ln(x^2+1) + c.$$

9°. **In general, if the integrand is a fraction in which the numerator is the derivative of the denominator, the indefinite integral is equal to the sum of the natural logarithm of the denominator and an arbitrary constant.**

Indeed, if we have

$$\int \frac{f'(x)}{f(x)}\,dx,$$

we put $f(x)=u$ and find that $f'(x)\,dx=du$:

$$\int \frac{f'(x)}{f(x)}\,dx=\int \frac{du}{u}=\ln u+c=\ln f(x)+c.$$

10°. Evaluate $\int \frac{x\,dx}{\sqrt{2x^2+5}}$.

Solution. In the fractional function $\frac{x}{\sqrt{2x^2+5}}$ the numerator x is of course not the derivative of the denominator. Hence one should not assume $\sqrt{2x^2+5}$ equal to u.

Noticing that the derivative of $2x^2+5$ is $4x$, we put $2x^2+5=u$ and take the differentials of both sides: $4x\,dx=du$, whence $x\,dx=\frac{1}{4}\,du$.

$$\int \frac{x\,dx}{\sqrt{2x^2+5}}=\int \frac{\frac{1}{4}\,du}{\sqrt{u}}=\frac{1}{4}\int u^{-\frac{1}{2}}\,du=\frac{1}{4}\,\frac{u^{\frac{1}{2}}}{\frac{1}{2}}+c=\frac{1}{2}\sqrt{u}+$$

$$+c=\frac{1}{2}\sqrt{2x^2+5}+c.$$

11°. Evaluate $\int \frac{(\ln x)^3}{x}\,dx$.

Solution. The integrand consists of two functions: a function of a function $(\ln x)^3$ and a simple function $\frac{1}{x}$. Let $\ln x=u$. Then $(\ln x)^3=u^3$ and $\frac{dx}{x}=du$. Hence,

$$\int \frac{(\ln x)^3\,dx}{x}=\int (\ln x)^3\cdot\frac{dx}{x}=\int u^3\,du=\frac{u^4}{4}+c=\frac{1}{4}(\ln x)^4+c.$$

12°. Evaluate $\int e^{x^4}\cdot x^3\,dx$.

Solution. Here e^{x^4} is a function of a function and x^3 is a simple function. Let $x^4=u$. We get $e^{x^4}=e^u$—a simple function of the variable u. Differentiating the equation $x^4=u$, we get $x^3\,dx=$ $=\frac{1}{4}\,du$. Then

$$\int e^{x^4}\cdot x^3\,dx=\int e^u\cdot\frac{1}{4}\,du=\frac{1}{4}\int e^u\,du=\frac{1}{4}e^u+c=\frac{1}{4}e^{x^4}+c.$$

13°. Let us examine a few instances of the simultaneous use of the methods of expansion and substitution.

When the integrand is an algebraic fraction, it is sometimes necessary to isolate the integral part of the function by dividing the numerator by the denominator according to the rules of division of polynomials.

For example,

$$\int \frac{4x+2}{2x-1}dx = \int \left(2+\frac{4}{2x-1}\right) dx = 2\int dx + 4\int \frac{dx}{2x-1}.$$

The first integral is tabular, the second can be found by substitution:

$$2x-1=u; \quad 2dx=du; \quad dx=\frac{1}{2}du.$$

$$2\int dx + 4\int \frac{dx}{2x-1} = 2x + 4\int \frac{\frac{1}{2}du}{u} = 2x + 2\int \frac{du}{u} =$$
$$= 2x + 2\ln u + c = 2x + 2\ln(2x-1) + c =$$
$$= 2x + \ln(2x-1)^2 + c.$$

14°. If the integrand is a product, it is sometimes useful to transform one of the factors without altering its value.

For example, $\int x\sqrt{x+1}\,dx$. In order to perform integration by substitution, add and subtract unity from the first factor x. We get

$$\int x\sqrt{x+1}\cdot dx = \int (x+1-1)\sqrt{x+1}\,dx =$$
$$= \int [(x+1)-1]\sqrt{x+1}\,dx = \int (x+1)\sqrt{x+1}\,dx -$$
$$- \int \sqrt{x+1}\,dx = \int (x+1)^{\frac{3}{2}}dx - \int (x+1)^{\frac{1}{2}}dx.$$

Both integrals are found by substitution.

$$x+1=u, \quad dx=du.$$

$$\int (x+1)^{\frac{3}{2}}dx - \int (x+1)^{\frac{1}{2}}dx = \int u^{\frac{3}{2}}du - \int u^{\frac{1}{2}}du =$$
$$= \frac{2}{5}u^2\sqrt{u} - \frac{2}{3}u\sqrt{u} + c = \frac{2}{15}u\sqrt{u}\,(3u-5) + c =$$
$$= \frac{2}{15}(x+1)\sqrt{x+1}\,(3x+3-5) + c =$$
$$= \frac{2}{15}(x+1)(3x-2)\sqrt{x+1} + c.$$

Sec. 121. Standard Integrals and Their Uses

1°. In solving examples in Sec. 120 we introduced a new variable u as a function of x and then employed the formulas of integration. Since this method is used very often, it is convenient to remember a number of formulas for the variable u regarding u

either as an argument or as the function of a different independent variable and taking du to be the differential of u.

Table of basic formulas.

I. $\int u^n\,du = \frac{u^{n+1}}{n+1} + c,\ n \neq -1.$ II. $\int \frac{du}{u} = \ln u + c.$

III. $\int e^u\,du = e^u + c.$ IV. $\int a^u\,du = \frac{a^u}{\ln a} + c.$

V. $\int \cos u\,du = \sin u + c.$ VI. $\int \sin u\,du = -\cos u + c.$

VII. $\int \frac{du}{\cos^2 u} = \tan u + c.$ VIII. $\int \frac{du}{\sin^2 u} = -\cot u + c.$

IX. $\int \frac{du}{1+u^2} = \arctan u + c.$ X. $\int \frac{du}{\sqrt{1-u^2}} = \arcsin u + c.$

The correctness of these formulas can be verified by differentiation.

2°. Let us study a few cases where formulas (V)-(VIII) are used.

1) $\int \cos 5x \cdot dx.$

Solution. $\cos 5x$ is a function of a function. To obtain a simple function and make use of formula (V) we put $5x = u$. Then $\cos 5x = \cos u,\ dx = \frac{1}{5}\,du.$

$$\int \cos 5x\,dx = \int \frac{1}{5}\cos u\,du = \frac{1}{5}\sin u + c = \frac{1}{5}\sin 5x + c.$$

2) $\int \frac{dx}{\sin^2 3x}$. To employ formula (VIII) put $3x = u$. Then $dx = \frac{1}{3}\,du.$

$$\int \frac{dx}{\sin^2 3x} = \frac{1}{3}\int \frac{du}{\sin^2 u} = -\frac{1}{3}\cot u + c = -\frac{1}{3}\cot 3x + c.$$

3) $\int x \cdot \sin(5x^2)\,dx$. Assume $5x^2 = u$. Then $x\,dx = \frac{1}{10}\,du.$

$$\int x \cdot \sin(5x^2)\,dx = \frac{1}{10}\int \sin u\,du = -\frac{1}{10}\cos u + c =$$
$$= -\frac{1}{10}\cos(5x^2) + c.$$

4) $\int \sin^3 x \cdot \cos x \cdot dx$. Here $\sin^3 x$ is a function of a function, and $\cos x$ is a simple function. Putting $\sin x = u$, we get $\sin^3 x = u^3$, which is a simple function of the variable u. Differentiating $\sin x = u$, we find that $\cos x\,dx = du.$

$$\int \sin^3 x \cos x\,dx = \int u^3\,du = \frac{u^4}{4} + c = \frac{1}{4}\sin^4 x + c.$$

5) $\int \frac{\sin\frac{x}{3}\,dx}{2+\cos\frac{x}{3}}$. If we put $2+\cos\frac{x}{3}=u$ and take the differentials of both sides, we obtain

$$-\frac{1}{3}\sin\frac{x}{3}\,dx=du, \quad \sin\frac{x}{3}\,dx=-3du,$$

$$\int \frac{\sin\frac{x}{3}\,dx}{2+\cos\frac{x}{3}}=-3\int\frac{du}{u}=-3\ln u+c=-3\ln\left(2+\cos\frac{x}{3}\right)+c.$$

6) $\int \frac{dx}{\sin x}$. Since $\sin x=2\sin\frac{x}{2}\cdot\cos\frac{x}{2}$,

$$\int\frac{dx}{\sin x}=\int\frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}}.$$

Multiplying the numerator and denominator of the fraction of $\cos\frac{x}{2}$ and knowing that

$$\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}=\frac{1}{\tan\frac{x}{2}}, \text{ we get}$$

$$\int\frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}}=\int\frac{\cos\frac{x}{2}\,dx}{2\sin\frac{x}{2}\cos^2\frac{x}{2}}=\int\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\cdot\frac{dx}{2\cos^2\frac{x}{2}}=$$

$$=\int\frac{1}{\tan\frac{x}{2}}\cdot\frac{dx}{2\cos^2\frac{x}{2}}.$$

Assume that $\tan\frac{x}{2}=u$ and take the differentials of both sides of this equation: $\frac{dx}{2\cos^2\frac{x}{2}}=du$. Hence

$$\int\frac{1}{\tan\frac{x}{2}}\cdot\frac{dx}{2\cos^2\frac{x}{2}}=\int\frac{1}{u}\,du=\ln u+c=\ln\tan\frac{x}{2}+c.$$

It is useful to remember that

$$\boxed{\int\frac{dx}{\sin x}=\ln\tan\frac{x}{2}+c}$$

7) $\int \frac{dx}{\cos x}$. This integral is reduced to the form of the previous integral by the use of the formula $\cos x = \sin\left(\frac{\pi}{2}+x\right)$.

Assuming $\frac{\pi}{2}+x=u$, we get $\cos x = \sin u$; $dx = du$. Consequently,

$$\int \frac{dx}{\cos x} = \int \frac{du}{\sin u} = \ln \tan \frac{u}{2} + c = \ln \tan\left(\frac{\pi}{4}+\frac{x}{2}\right)+c.$$

It is useful to remember that

$$\boxed{\int \frac{dx}{\cos x} = \ln \tan\left(\frac{\pi}{4}+\frac{x}{2}\right)+c}$$

3°. Let us examine the application of the basic formulas (IX) and (X).

1. Reduce $\int \frac{dx}{\sqrt{25-16x^2}}$ to the tabular expression $\int \frac{du}{\sqrt{1-u^2}}$ by substitution of $16x^2=25u^2$.

Whence

$$4x=5u, \quad dx=\frac{5}{4}\,du.$$

$$\int \frac{dx}{\sqrt{25-16x^2}} = \frac{5}{4}\int \frac{du}{\sqrt{25-25u^2}} = \frac{5}{4}\int \frac{dx}{5\sqrt{1-u^2}} =$$
$$= \frac{1}{4}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{4}\ \text{arc sin}\, u + c = \frac{1}{4}\ \text{arc sin}\, \frac{4}{5}\, x + c,$$

since from $4x=5u$ it follows that $u=\frac{4}{5}\,x$.

2) Reduce $\int \frac{dx}{5+3x^2}$ to the tabular form $\int \frac{du}{1+u^2}$ by substitution of $3x^2=5u^2$.

Whence $x\sqrt{3}=u\sqrt{5}$; $dx=\frac{\sqrt{5}}{\sqrt{3}}\,du$.

$$\int \frac{dx}{5+3x^2} = \frac{\sqrt{5}}{\sqrt{3}}\int \frac{du}{5+5u^2} = \frac{\sqrt{5}}{5\sqrt{3}}\int \frac{du}{1+u^2} = \frac{1}{\sqrt{15}}\int \frac{du}{1+u^2} =$$
$$= \frac{1}{\sqrt{15}}\ \text{arc tan}\, u + c = \frac{1}{\sqrt{15}}\ \text{arc tan}\sqrt{\frac{3}{5}}\, x + c.$$

Since from the equation $x\sqrt{3}=u\cdot\sqrt{5}$ it follows that

$$u=\sqrt{\frac{3}{5}}\cdot x.$$

3) Reduce $\int \frac{dx}{\sqrt{3-5x^2}}$ to $\int \frac{du}{\sqrt{1-u^2}}$ by substitution of $5x^2=3u^2$.

Whence $x\sqrt{5}=u\sqrt{3}$, $dx=\frac{\sqrt{3}}{\sqrt{5}}\,du$.

$$\int\frac{dx}{\sqrt{3-5x^2}}=\frac{\sqrt{3}}{\sqrt{5}}\int\frac{du}{\sqrt{3-3u^2}}=\frac{\sqrt{3}}{\sqrt{5}}\int\frac{du}{\sqrt{3}\cdot\sqrt{1-u^2}}=$$

$$=\frac{1}{\sqrt{5}}\int\frac{du}{\sqrt{1-u^2}}=\frac{1}{\sqrt{5}}\arcsin\sqrt{\frac{5}{3}}\,x+c,$$

since it follows from the equality $x\sqrt{5}=u\sqrt{3}$ that $u=\sqrt{\frac{5}{3}}\cdot x$.

4) Reduce $\int\frac{\cos x\,dx}{\sqrt{2-\sin^2 x}}$ to $\int\frac{du}{\sqrt{1-u^2}}$ by substitution of $\sin^2 x=$ $=2u^2$.

Whence $\sin x=u\sqrt{2}$, $\cos x\,dx=\sqrt{2}\cdot du$.

$$\int\frac{\cos x\,dx}{\sqrt{2-\sin^2 x}}=\sqrt{2}\int\frac{du}{\sqrt{2-2u^2}}=\sqrt{2}\int\frac{du}{\sqrt{2}\cdot\sqrt{1-u^2}}=$$

$$=\int\frac{du}{\sqrt{1-u^2}}=\arcsin u+c=\arcsin\frac{\sin x}{\sqrt{2}}+c,$$

since equation $\sin x=u\sqrt{2}$ gives $u=\frac{\sin x}{\sqrt{2}}$.

5) Write $\int\frac{dx}{x(4+\ln^2 x)}$ as $\int\frac{1}{4+(\ln x)^2}\cdot\frac{dx}{x}$ and reduce to tabular form $\int\frac{du}{1+u^2}$ by substitution of $(\ln x)^2=4u^2$.

Whence $\ln x=2u$, $\frac{dx}{x}=2du$.

$$\int\frac{1}{4+(\ln x)^2}\cdot\frac{dx}{x}=2\int\frac{du}{4+4u^2}=\frac{1}{2}\int\frac{du}{1+u^2}=\frac{1}{2}\arctan u+c=$$

$$=\frac{1}{2}\arctan\frac{\ln x}{2}+c=\frac{1}{2}\arctan\ln\sqrt{x}+c,$$

since equation $\ln x=2u$ gives $u=\frac{\ln x}{2}$.

4°. It is also useful to know the following two formulas:

$$\int\frac{du}{1-u^2}=\frac{1}{2}\ln\frac{1+u}{1-u}+c. \qquad \text{(XI)}$$

$$\int\frac{du}{\sqrt{u^2\pm 1}}=\ln\left(u+\sqrt{u^2\pm 1}\right) \qquad \text{(XII)}$$

Let us examine a few instances where these formulas are applied.

1) Reduce $\int\frac{dx}{\sqrt{4x^2-3}}$ to the tabular form $\int\frac{du}{\sqrt{u^2-1}}$ by substitution of $4x^2=3u^2$.

Whence $2x=u\sqrt{3}$, $dx=\frac{\sqrt{3}}{2}\,du$.

$$\int\frac{dx}{\sqrt{4x^2-3}}=\frac{\sqrt{3}}{2}\int\frac{du}{\sqrt{3u^2-3}}=\frac{1}{2}\int\frac{du}{\sqrt{u^2-1}}=$$

$$=\frac{1}{2}\ln\left(u+\sqrt{u^2-1}\right)+c=\frac{1}{2}\ln\left(\frac{2x}{\sqrt{3}}+\sqrt{\frac{4x^2}{3}-1}\right)+c=$$

$$=\frac{1}{2}\ln\frac{2x+\sqrt{4x^2-3}}{\sqrt{3}}+c=\frac{1}{2}\ln\left(2x+\sqrt{4x^2-3}\right)+$$

$$+c-\ln\sqrt{3}=\frac{1}{2}\ln\left(2x+\sqrt{4x^2-3}\right)+c,$$

since $\ln\sqrt{3}$, as a constant, can be included in c.

2) $\int\frac{x+4}{4x^2-5}\,dx$ may be written as a sum of integrals by dividing the numerator by the denominator termwise:

$$\int\frac{x+4}{4x^2-5}\,dx=\int\frac{x\,dx}{4x^2-5}+4\int\frac{dx}{4x^2-5}=\int\frac{x\,dx}{4x^2-5}-4\int\frac{dx}{5-4x^2}.$$

The first integral is found by substitution of

$$4x^2-5=u,\ 8x\,dx=du,\ x\,dx=\frac{1}{8}\,du.$$

$$\int\frac{x\,dx}{4x^2-5}=\frac{1}{8}\int\frac{du}{u}=\frac{1}{8}\ln u+c=\frac{1}{8}\ln(4x^2-5)+c.$$

The second integral is reduced to the tabular form $\int\frac{du}{1-u^2}$ by substitution of $4x^2=5t^2$.

Whence $2x=t\sqrt{5}$, $dx=\frac{\sqrt{5}}{2}\,dt$.

$$\int\frac{dx}{5-4x^2}=\frac{\sqrt{5}}{2}\int\frac{dt}{5-5t^2}=\frac{\sqrt{5}}{5\cdot 2}\int\frac{dt}{1-t^2}=\frac{1}{2\sqrt{5}}\cdot\frac{1}{2}\ln\frac{1+t}{1-t}+c=$$

$$=\frac{1}{4\sqrt{5}}\ln\frac{1+\frac{2}{\sqrt{5}}x}{1-\frac{2}{\sqrt{5}}x}+c=\frac{1}{4\sqrt{5}}\ln\frac{\sqrt{5}+2x}{\sqrt{5}-2x}+c,$$

since $t=\frac{2}{\sqrt{5}}\,x$.

Finally,

$$\int\frac{x+4}{4x^2-5}\,dx=\frac{1}{8}\ln(4x^2-5)-\frac{1}{4\sqrt{5}}\ln\frac{\sqrt{5}+2x}{\sqrt{5}-2x}+c.$$

Sec. 122. Integration of Powers of $\sin x$, $\cos x$, $\tan x$, $\cot x$

In this section the exponents are considered positive integers.

1°. $\int \sin^m x\,dx$ and $\int \cos^m x\,dx$ where m is an *odd number*.

For example, evaluate $\int \sin^3 x\,dx$.

Solution. First expand the integrand:

$$\sin^3 x=\sin^2 x\cdot\sin x=(1-\cos^2 x)\cdot\sin x=\sin x-\cos^2 x\cdot\sin x.$$

Then

$$\int \sin^3 x\,dx=\int \sin x\,dx-\int \cos^2 x\cdot\sin x\cdot dx.$$

The first integral is tabular; the second is found by substitution:

$$\cos x=u,\ \cos^2 x=u^2,\ \sin x\,dx=-du.$$

$$\int \sin x\,dx-\int \cos^2 x\sin x\,dx=-\cos x+\int u^2\,du=\frac{u^3}{3}-$$

$$-\cos x+c=\frac{\cos^3 x}{3}-\cos x+c.$$

2°. $\int \sin^m x\,dx$ and $\int \cos^m x\,dx$, where m is an *even number*, are found by familiar formulas of trigonometry:

$$1-\cos\alpha=2\sin^2\frac{\alpha}{2}\ \text{and}\ 1+\cos\alpha=3\cos^2\frac{\alpha}{2},$$

which are taken in the following form:

$$\text{(XIII)}\qquad \boxed{\sin^2 x=\frac{1-\cos 2x}{2};}\qquad\qquad \boxed{\cos^2 x=\frac{1+\cos 2x}{2}}\qquad \text{(XIV)}$$

These formulas reduce the power of the function.

Example 1. Evaluate $\int \sin^2 x\,dx$.

Solution. Using formula (XIII),

$$\int \sin^2 x\,dx=\int \frac{1-\cos 2x}{2}\,dx=\frac{1}{2}\int dx-\frac{1}{2}\int \cos 2x\,dx.$$

The first integral is tabular; the second is found by substitution of $2x=u$, $dx=\frac{1}{2}\,du$.

Substituting, we get

$$\int \sin^2 x\,dx=\frac{1}{2}\int dx-\frac{1}{2}\int \cos 2x\,dx=\frac{1}{2}x-\frac{1}{4}\int \cos u\,du=$$

$$=\frac{1}{2}x-\frac{1}{4}\sin u+c=\frac{1}{2}x-\frac{1}{4}\sin 2x+c.$$

Example 2. Evaluate $\int \cos^4 x\,dx$.

Solution. First expand the function $\cos^4 x$ using formula (XIV):

$$\cos^4 x=(\cos^2 x)^2=\left(\frac{1+\cos 2x}{2}\right)^2=\frac{1}{4}\,(1+2\cos 2x+\cos^2 2x).$$

Again applying formula (XIV) to $\cos^2 2x$, we get

$$\frac{1}{4}+\frac{1}{2}\cos 2x+\frac{1}{4}\,\frac{1+\cos 4x}{2}=\frac{1}{4}+\frac{1}{2}\cos 2x+\frac{1}{8}+\frac{1}{8}\cos 4x=$$
$$=\frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x.$$

Hence

$$\int \cos^4 x\,dx=\frac{3}{8}\int dx+\frac{1}{2}\int \cos 2x\,dx+\frac{1}{8}\int \cos 4x\,dx.$$

Assuming

$$2x=u,\quad dx=\frac{1}{2}\,du,\quad 4x=t,\quad dx=\frac{1}{4}\,dt,$$

we get

$$\frac{3}{8}\int dx+\frac{1}{4}\int \cos u\,du+\frac{1}{32}\int \cos t\,dt=\frac{3}{8}\,x+\frac{1}{4}\sin u+\frac{1}{32}\sin t+c=$$
$$=\frac{3}{8}\,x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+c.$$

3°. $\int \tan^m x\,dx$ and $\int \cot^m x\,dx$ are found by the successive use of the trigonometric formulas:

$$\boxed{\tan^2 x=\sec^2 x-1}\quad \text{(XV)}\quad \text{and}\quad \boxed{\cot^2 x=\operatorname{cosec}^2 x-1}\quad \text{(XVI)}$$

The transformation is done with a view to getting integrals of the form:

$$\int \tan^n x\cdot\sec^2 x\cdot dx \quad\text{and}\quad \int \cot^n x\cdot\operatorname{cosec}^2 x\cdot dx,$$

which are then solved by the following substitutions:

$$\tan x=u,\qquad \sec^2 x\cdot dx=du,$$
$$\cot x=u,\qquad \operatorname{cosec}^2 x\cdot dx=-du.$$

Example. Evaluate $\int \tan^3 x\cdot dx$.

Solution. Transform $\tan^3 x$ using formula (XVI):

$$\tan^3 x=\tan x\cdot\tan^2 x=\tan x\,(\sec^2 x-1)=\tan x\cdot\sec^2 x-\tan x=$$
$$=\tan x\sec^2 x-\frac{\sin x}{\cos x}.$$

Hence

$$\int \tan^3 x\,dx=\int \tan x\cdot\sec^2 x\cdot dx-\int \frac{\sin x\,dx}{\cos x}.$$

Putting $\tan x=u$, $\sec^2 x\,dx=du$; $\cos x=t$, $\sin x\,dx=-dt$, we obtain

$$\int \tan^3 x\,dx=\int u\,du-\int \frac{dt}{t}=\frac{1}{2}\,u^2-\ln t+c=\frac{1}{2}\tan^2 x-\ln\cos x+c.$$

Sec. 123. $\int \sqrt{a^2-x^2}\,dx$

$\int \sqrt{a^2-x^2}\,dx$ is frequently encountered in practice and is evaluated by substituting $x=a\cdot\sin\varphi$. This is called trigonometric substitution.

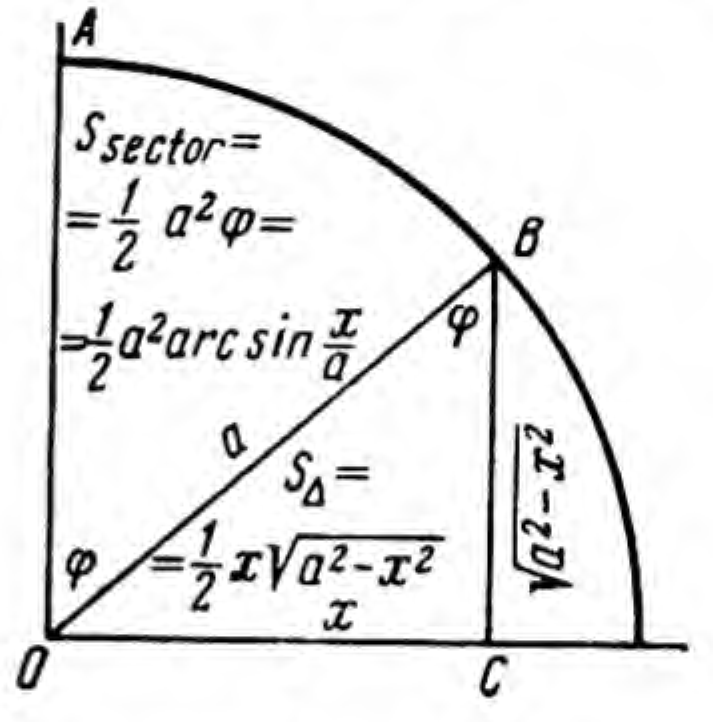

Fig. 126.

Evaluate $\int \sqrt{a^2-x^2}\,dx$. Putting $x=$ $=a\cdot\sin\varphi$, we have

$$x^2=a^2\cdot\sin^2\varphi,\ dx=a\cdot\cos\varphi\,d\varphi,$$

$$\int \sqrt{a^2-x^2}dx=$$

$$=\int \sqrt{a^2-a^2\cdot\sin^2\varphi}\cdot a\cdot\cos\varphi\,d\varphi=$$

$$=a^2\int \sqrt{1-\sin^2\varphi}\cdot\cos\varphi\,d\varphi=$$

$$=a^2\int \cos^2\varphi\cdot d\varphi=$$

$$=a^2\cdot\int \frac{1+\cos 2\varphi}{2}\,d\varphi=\frac{1}{2}a^2\int d\varphi+\frac{1}{2}a^2\int \cos 2\varphi\cdot d\varphi=$$

$$=\frac{1}{2}a^2\varphi+\frac{1}{4}a^2\sin 2\varphi+c.$$

$\int \cos 2\varphi\cdot d\varphi$ is found by substitution of $2\varphi=t$, $d\varphi=\frac{1}{2}\,dt$.

Let us find the values of φ and $\sin 2\varphi$.

Since $x=a\cdot\sin\varphi$, $\sin\varphi=\frac{x}{a}$, and $\varphi=\arcsin\frac{x}{a}$.

$$\sin 2\varphi=2\sin\varphi\cdot\cos\varphi=2\sin\varphi\sqrt{1-\sin^2\varphi}=$$

$$=2\cdot\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}=\frac{2x}{a^2}\sqrt{a^2-x^2}.$$

These values of $\varphi=\arcsin\frac{x}{a}$ and $\sin 2\varphi=\frac{2x}{a^2}\sqrt{a^2-x^2}$ are put into the integral:

$$\int \sqrt{a^2-x^2}\,dx=\frac{1}{2}a^2\varphi+\frac{1}{4}a^2\sin 2\varphi+c=$$

$$=\frac{1}{2}a^2\arcsin\frac{x}{a}+\frac{a^2}{4}\frac{2x}{a^2}\sqrt{a^2-x^2}+c,$$

whence

$$\boxed{\int \sqrt{a^2-x^2}\,dx=\frac{1}{2}a^2\arcsin\frac{x}{a}+\frac{1}{2}x\cdot\sqrt{a^2-x^2}+c} \qquad \text{(XVII)}$$

To make it easier to remember this formula, let us interpret its terms geometrically: the first term of the formula $\frac{1}{2}a^2 \arcsin \frac{x}{a} = \frac{1}{2}a^2\varphi$ is the area of the sector AOB (Fig. 126) with radius a and arc φ, the second term $\frac{1}{2}x\sqrt{a^2-x^2}$ is the area of a triangle with hypotenuse a and one leg x lying opposite the angle φ.

Sec. 124. Remarks

To conclude this chapter on the indefinite integral it may be called to mind once more that one of the problems of integral calculus is to find the antiderivative from the given derivative of $f(x)$, and that if the given function $f(x)$ is continuous on the interval $a \leqslant x \leqslant b$, there exists for it an antiderivative. But if an antiderivative exists for every continuous function, is it always possible to find this antiderivative?

Analysis gives a direct answer to this question: not every indefinite integral can be expressed in terms of algebraic, trigonometric, inverse circular, logarithmic or exponential functions as a result of elementary operations on them.

For example, $\int \frac{dx}{\ln x}$ cannot be found as a result of elementary mathematical operations. But the values of this integral can be found approximately to any desired degree of accuracy by means of an infinite series.

Integral calculus provides methods for finding integrals of only certain (true, the number is rather large) functions.

We considered only the very simplest methods of integration and as applied only to the simplest functions.

CHAPTER XI

THE DEFINITE INTEGRAL AND ITS APPLICATIONS

Sec. 125. The Definite Integral as a Measure of the Amount of Change in the Antiderivative

Let us examine how to find the amount of change of a function with respect to its derivative $f(x)$ as the argument x varies from $x=a$ to $x=b$. We assume that the given function $f(x)$ is continuous in the interval $[a, b]$ where $a<b$.

1°. **Theorem.** *When the argument x varies from $x=a$* to *$x=b$, each of the antiderivatives of the given function $f(x)$ has one and the same increment.*

Proof. From a set of antiderivatives whose derivative is $f(x)$ let us take any two of them and denote them by $F(x)$ and $\Phi(x)$. They will differ from each other by a certain number c:

$$F(x)-\Phi(x)=c.$$

Whence

$$F(x)=\Phi(x)+c.$$

Let us determine the values of these two antiderivatives for $x=b$ and $x=a$.

$$\text{For } x=b, \quad F(b)=\Phi(b)+c.$$
$$\text{For } x=a, \quad F(a)=\Phi(a)+c.$$

Subtracting the second equality from the first, we get $F(b)-F(a)=\Phi(b)-\Phi(a)$, as required.

2°. **Definition.** *The difference $F(b)-F(a)$—which is the value of the increment of any antiderivative of the given function $f(x)$ as the argument x varies from $x=a$ to $x=b$—is called the definite integral of the function $f(x)$ between the limits a and b and is represented by the symbol*

$$\int_a^b f(x)\,dx.$$

The numbers a and b are called, respectively, the lower and upper limits of the definite integral. Of course the numbers a and

b are not limits in the sense in which we have hitherto used the word. They are merely the boundaries of the range of variation of x.

The symbol $\int_a^b f(x)\,dx$ stands for "the definite integral from a to b of the function $f(x)\,dx$". By definition

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

3°. **Rule.** *To evaluate a definite integral it is sufficient to:*

1) *find the indefinite integral of the given function;*

2) *take the functional part of the indefinite integral and substitute into it, in place of x, first the upper limit b and then the lower limit a; and then subtract the second resultant value of the substitution from the first.*

The second operation is symbolically written as $[F(x)]_a^b$ or $F(x)\Big|_a^b$ or $\Big|_a^b F(x)$.

In all cases we read "the value of $F(x)$ substituted from a to b". We shall use the symbol $\Big|_a^b F(x)$.

The rule for evaluating a definite integral is written symbolically as

$$\boxed{\int_a^b f(x)\,dx = \Big|_a^b F(x) = F(b) - F(a)} \qquad \text{(XVIII)}$$

Let us note that in this formula the function $f(x)$ stands under the integral sign $\left(\int_a^b\right)$ while the antiderivative $F(x)$, the functional part of the indefinite integral, stands under the sign of substitution $\left(\Big|_a^b\right)$.

Example. To evaluate $\int_1^2 x^3\,dx$,

1) find the indefinite integral: $\int x^3\,dx = \frac{x^4}{4} + c$;

2) the functional part is $\frac{x^4}{4}$, the values of which on substitution from 1 to 2 are

$$\left.\frac{x^4}{4}\right|_1^2=\frac{2^4}{4}-\frac{1^4}{4}=\frac{15}{4}=3\,\frac{3}{4}.$$

4°. **Examples.** 1. The slope of the tangent at every point of a curve is equal to $2x$. How does the ordinate of a point on this curve vary when the abscissa changes from 2 to 3?

Solution. It is given that

$$\tan\varphi=\frac{dy}{dx}=2x.$$

Therefore the ordinate y is an antiderivative of $2x$:

$$y=\int 2x\,dx,$$

or

$$y=x^2+c.$$

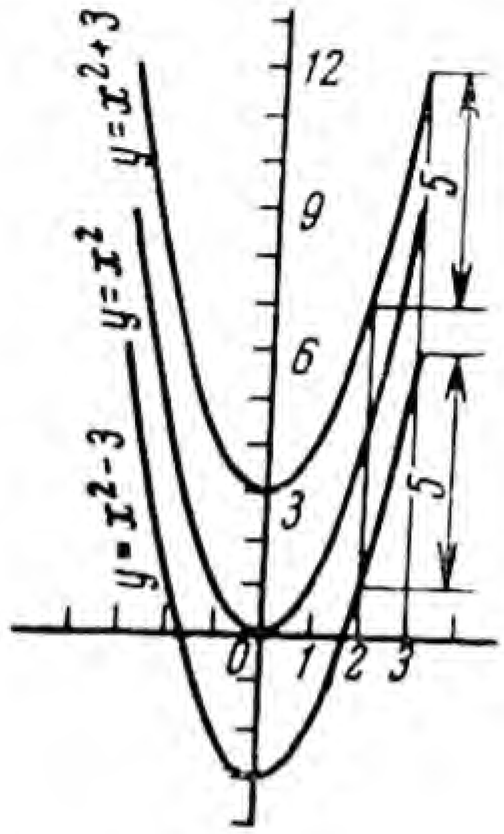

Fig. 127.

Figure 127 illustrates these curves for different values of c: -3, 0, $+3$.

As x changes from 2 to 3, the ordinate y changes in value by $\int_2^3 2x\,dx$.

$$\int_2^3 2x\,dx=\left.x^2\right|_2^3=3^2-2^2=5.$$

In Fig. 127 this increment in the ordinate, equal to 5, is shown on the different curves: $y=x^2-3$, $y=x^2$, $y=x^2+3$.

2. Determine the distance travelled by a point from time $t=3$ min to $t=9$ min if the velocity at every instant of time is t^2 m/min.

Solution. The velocity at any instant is a derivative of the distance with respect to time, $\frac{ds}{dt}$. It is given that

$$\frac{ds}{dt}=t^2.$$

The distance s is an antiderivative of t^2. And the distance s covered between $t=3$ min and $t=9$ min is the amount of change in the antiderivative of t^2 as the argument t changes from 3 to 9.

The distance is $\int_3^9 t^2\,dt$. Evaluating this expression, we have $\int_3^9 t^2\,dt$.

$$1)\quad \int t^2\,dt = \frac{t^3}{3} + c;$$

$$2)\quad \int_3^9 t^2\,dt = \Big|_3^9 \frac{t^3}{3} = \frac{9^3}{3} - \frac{3^3}{3} = 243 - 9 = 234.$$

Consequently, $s = 234$ m.

3. The heat capacity of 1 kg of water varies with the temperature t according to the law $f(t) = 1 + 0.00004\,t + 0.0000009\,t^2$. Determine the amount of heat required to raise 1 m³ of water from 10°C to 60°C.

Solution. For 1 m³ of water the heat capacity

$$f(t) = 1000 + 0.04t + 0.0009t^2.$$

The heat capacity is the derivative of the amount of heat with respect to temperature, $\frac{dQ}{dt}$:

$$\frac{dQ}{dt} = 1000 + 0.04t + 0.0009t^2.$$

For the given function $f(t)$, the quantity of heat, Q, is the antiderivative. It is required to find out how much the antiderivative changes as t changes from 10° to 60°, i.e., to evaluate the definite integral

$$\int_{10}^{60} (1000 + 0.04t + 0.0009t^2)\,dt.$$

First evaluate the indefinite integral:

$$\int (1000 + 0.04t + 0.0009t^2)\,dt = 1000t + 0.02t^2 + 0.0003t^3 + c$$

and then its value on substitution from 10 to 60:

$$\Big|_{10}^{60} (1000t + 0.02t^2 + 0.0003t^3) = (60000 + 72 + 64.8) -$$
$$-(10000 + 2 + 0.3) = 50134.5.$$

Thus, 50134.5 kcal of heat is required to raise the temperature of 1 m³ of water from 10° to 60°.

Sec. 126. The Definite Integral as a Function

1°. In the definite integral $\int_a^b f(x)\,dx = F(b) - F(a)$, the limits a and b are certain definite values of x such that on the interval $[a, b]$ the given function $f(x)$ is continuous. Let us keep the lower limit a constant and consider the upper limit b variable and denote it by x.

$$\int_a^x f(x)\,dx = F(x) - F(a).$$

Since to every value of x there corresponds a certain definite number, the difference $F(x) - F(a)$, *the definite integral* $\int_a^x f(x)\,dx$ *is a function of its upper limit* x.

2°. From the set of antiderivatives $F(x) + c$ whose derivative is $f(x)$, let us find the one which is equal to zero at $x = a$.

Substituting a for x in $F(x) + c$, we obtain

$$F(a) + c = 0,$$

$$c = -F(a).$$

Whence it follows that the required antiderivative is $F(x) - F(a)$. But

$$F(x) - F(a) = \int_a^x f(x)\,dx.$$

Consequently, *the definite integral* $\int_a^x f(x)\,dx$ *is that particular antiderivative of* $f(x)$ *which becomes zero at* $x = a$.

3°. **Relationship between the indefinite and definite integral.** Since $\int_a^x f(x)\,dx$ is one particular antiderivative of $f(x)$, namely, that one which becomes zero at $x = a$, every other antiderivative differs from this one by an arbitrary constant c. In other words, an indefinite integral differs from a definite integral by an arbitrary constant c:

$$\int f(x)\,dx = \int_a^x f(x)\,dx + c.$$

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1°. Figure 128 shows the graph of a continuous positive function $y = f(x)$. Let us take two points A and M on this curve of $y = f(x)$, and let us treat A as fixed and M as moving. Accordingly, the coordinates of A are the constants $[a, f(a)]$ and those of M are the variables (x, y). To every definite value of x there corresponds a definite area s of the curvilinear trapezoid A_1AMM_1 bounded by the arc AM, the axis Ox and the ordinates A_1A and M_1M of the points A and M. Hence, the area s is a function of x, a positive function since all values of s are positive.

We shall show that *s is an antiderivative of the given function $f(x)$*. To do this, find the derivative of s with respect to x. Let x have some definite value, then add to it an increment Δx. The area s thereby receives an increment Δs equal to the area of the figure M_1MNN_1.

Let the function $y = f(x)$ increase and decrease. Then on the segment M_1N_1 of the change Δx there will be found a least ordinate B_1B and a greatest ordinate C_1C*. Let us construct two rectangles $M_1KK_1N_1$ and $M_1LL_1N_1$ using Δx as the base in each case and B_1B and C_1C as their respective altitudes. We get area $M_1KK_1N_1 < \Delta s <$ $<$ area $M_1LL_1N_1$, since the first area is a part of the second and the second is a part of the third area.

But area $M_1KK_1N_1 = \Delta x \cdot B_1B$ and area $M_1LL_1N_1 = \Delta x \cdot C_1C$. Hence

$$\Delta x \cdot B_1B < \Delta s < \Delta x \cdot C_1C.$$

Dividing through by Δx, we get

$$B_1B < \frac{\Delta s}{\Delta x} < C_1C.$$

Let Δx tend to zero. As $\Delta x \to 0$ the variable ordinates B_1B and C_1C—since the curve is continuous—approach the constant ordinate $M_1M = y$ as their limit. Hence (Sec. 56)

$$\lim_{\Delta x \to 0} \frac{\Delta s}{\Delta x} = M_1M,$$

or

$$\frac{ds}{dx} = y.$$

In words, *the derivative of the area s with respect to the abscissa x is equal to the ordinate y of the curve at the point x.*

But since $y = f(x)$,

$$\frac{ds}{dx} = f(x).$$

* This is proved in advanced courses of analysis.

It follows from this that *the area s bounded by the graph of the function $y = f(x)$, the axis of abscissas and the ordinates of two points on the graph is an antiderivative of the given function $f(x)$.*

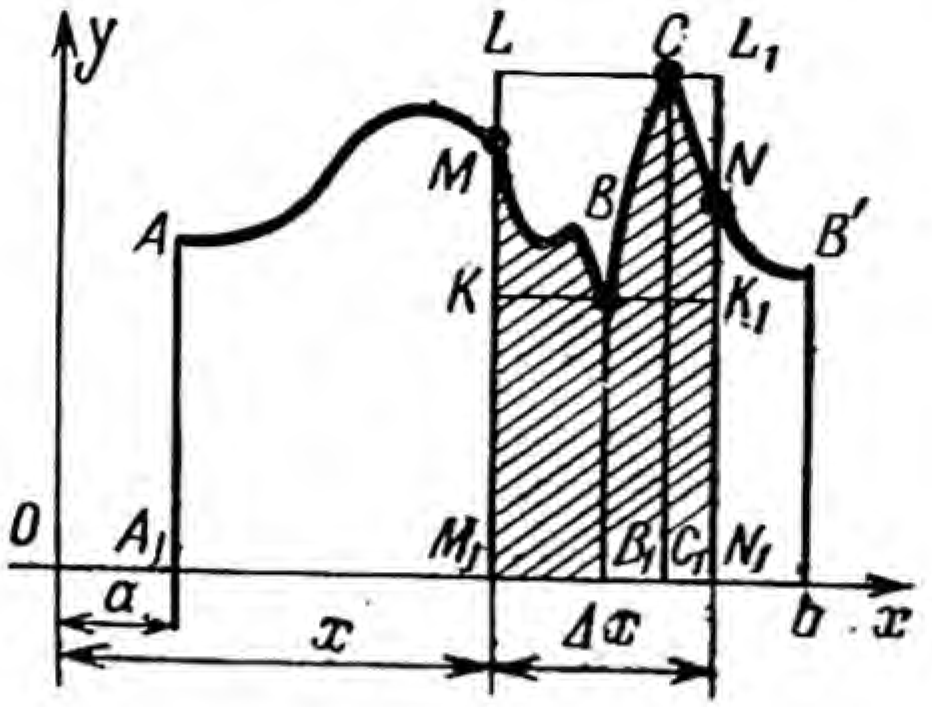

Fig. 128.

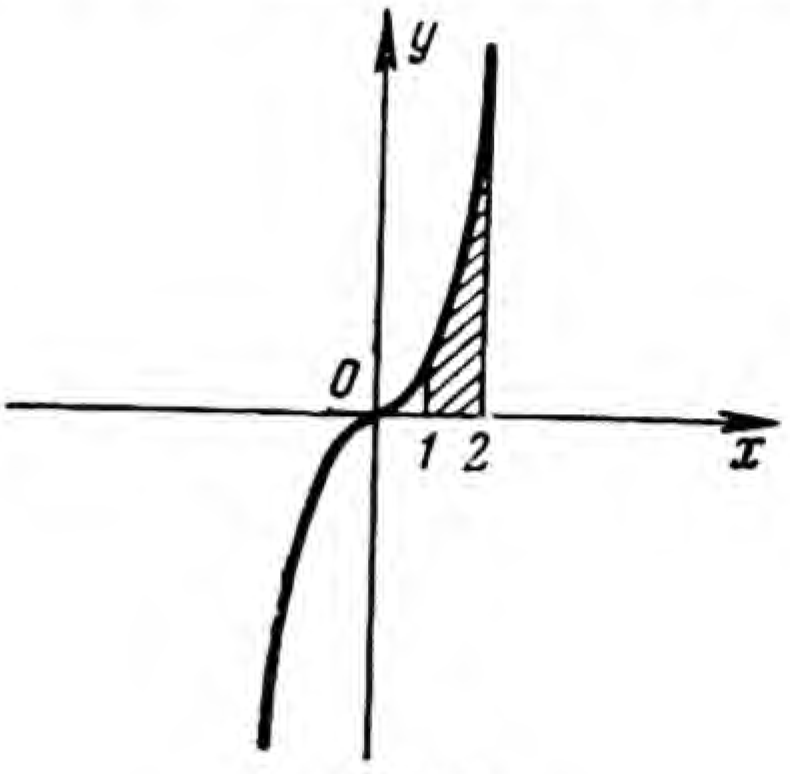

Fig. 129.

The area A_1AMM_1 is that antiderivative of $f(x)$ which becomes zero at $x = a$. Indeed, if the moving point M coincides with the fixed point A (which occurs at $x = a$), the area A_1AMM_1 becomes zero. Hence (Sec. 126)

$$s = \int_a^x f(x)\,dx.$$

To obtain the value of the area $A_1AB'b$ (Fig. 128) enclosed between the ordinates $x = a$ and $x = b$ it is sufficient to replace x by its value b. Then we get

$$\boxed{s = \int_a^b f(x)\,dx} \qquad \text{(XIX)}$$

This is *the geometrical meaning of a definite integral: the definite integral $\int_a^b f(x)\,dx$ of a continuous positive function $f(x)$ is equal to the value of the area enclosed between the curve $y = f(x)$, the axis of abscissas Ox and two ordinates $x = a$ and $x = b$.*

2°. **Example.** Calculate the area enclosed between the curve $y = x^3$, the axis Ox and the ordinates $x = 1$ and $x = 2$ (Fig. 129).

Solution. Basic formula for area:

$$s = \int_a^b y \cdot dx.$$

It is given that $y=x^3$, $a=1$, $b=2$. Therefore

$$s=\int_1^2 x^3\,dx=\Big|_1^2 \frac{x^4}{4}=\frac{2^4}{4}-\frac{1^4}{4}=3\,\frac{3}{4}.$$

3°. **Note.** Sometimes it is wrongly asserted that "the definite integral is an area". This is erroneous because the definite integral $\int_a^b f(x)\,dx$ is a number which may represent quite different kinds of quantities: in some instances, areas, in others (as was shown in some previous examples) increments in ordinates, distances covered by a moving body, quantities of heat, and so forth. Further on we shall see that the definite integral $\int_a^b f(x)\,dx$ can represent the volume of a body, work, pressure and other specific quantities.

Sec. 128. Supplementary Notes

1°. In the preceding section it was shown that the area enclosed by the curve $y=f(x)$, the ordinates of two of its points, and the axis Ox is an antiderivative of $f(x)$ if $f(x)$ is a positive function $[f(x)>0]$ on the interval $a\leqslant x\leqslant b$, i.e., if the curve $y=f(x)$ lies in the upper half of the plane divided by the axis Ox. This also holds if $f(x)$ is negative on the interval $[a, b]$ $[f(x)<0]$, i.e., if the line $y=f(x)$ lies in the lower half of the plane relative to the axis Ox.

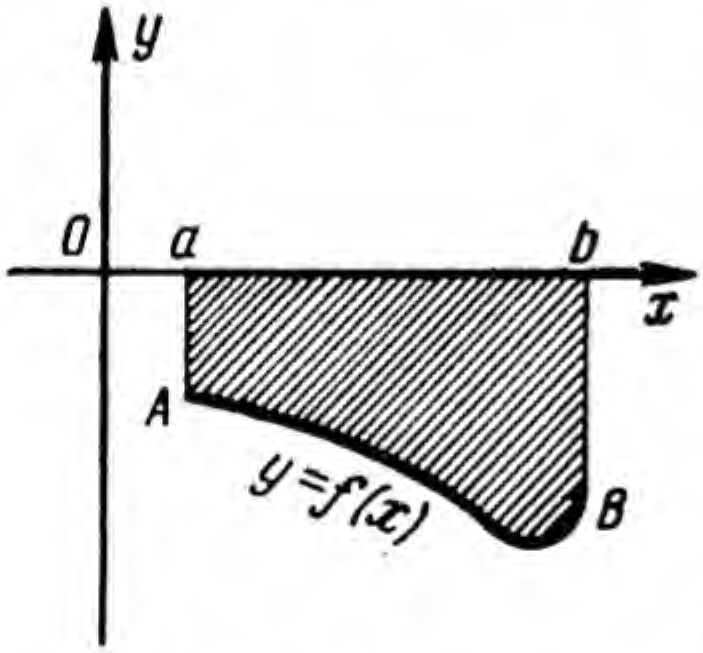

Fig. 130.

If the function has negative values, the foregoing reasoning will apply in obtaining the expression for s, the area bounded by the curve $y=f(x)$, the axis Ox and the ordinates $x=a$ and $x=b$ (Fig. 130), the only difference being that the value of the area lying below the axis Ox has to be treated as negative since the ordinates of points of the curve are negative:

$$\frac{ds}{dx}=y<0$$

and $\int_a^b f(x)\,dx$ is negative.

The sign of $\int_a^b f(x)\,dx$ only indicates the position of the area relative to the axis Ox: plus, signifying "above the axis" and minus, "below the axis". The actual value of the area is equal to the absolute value of the integral.

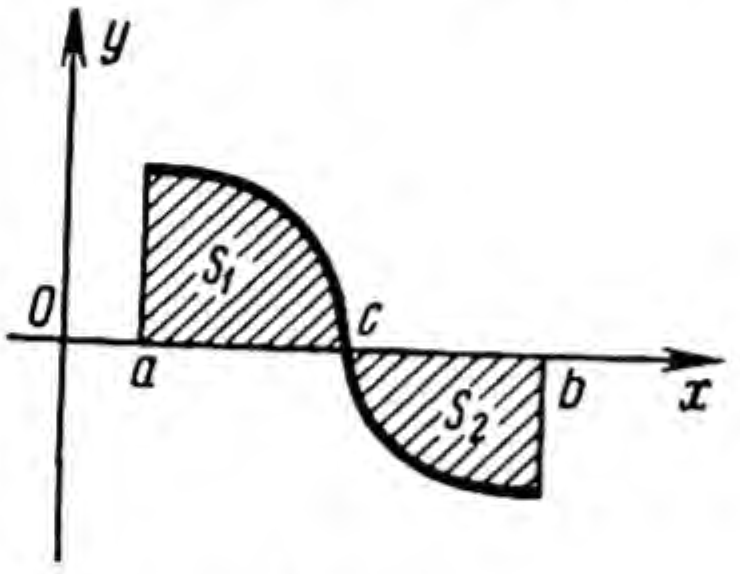

Fig. 131.

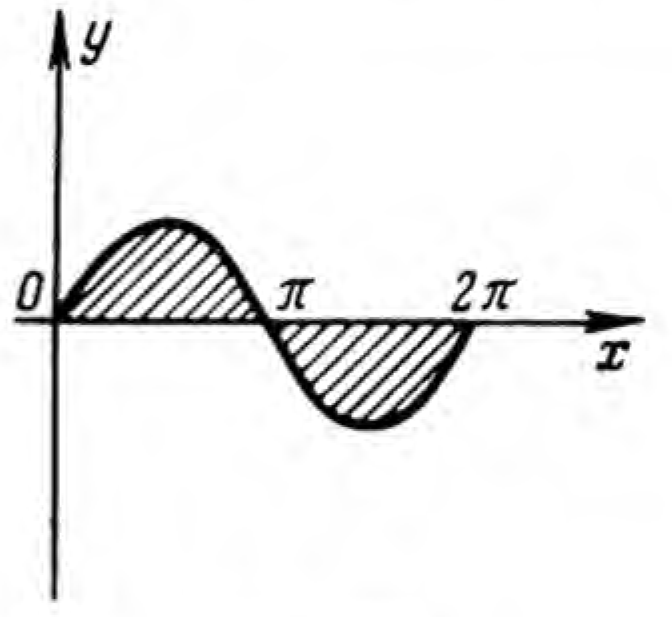

Fig. 132.

If the area lies partly above and partly below the x-axis (Fig. 131), the entire area is obtained by evaluating separately the area s_1 above the x-axis and the area s_2 below the x-axis:

$$s = |s_1| + |s_2|.$$

Thus the entire area is equal to the sum of the absolute values of the areas of its parts lying above and below the x-axis.

Example. Calculate the area bounded by a sinusoidal wave $y = \sin x$ and the x-axis (Fig. 132).

Solution. One part of this area ($0 \leqslant x \leqslant \pi$) lies above the x-axis, the other part ($\pi \leqslant x \leqslant 2\pi$), below it. Calculating them separately, first part

$$s_1 = \int_0^{\pi} \sin x\,dx = \Big|_0^{\pi} -\cos x = -\cos\pi + \cos 0 = 1 + 1 = 2,$$

second part

$$s_2 = \int_{\pi}^{2\pi} \sin x\,dx = \Big|_{\pi}^{2\pi} -\cos x = -\cos 2\pi + \cos\pi = -1 - 1 = -2.$$

The whole area $s = |s_1| + |s_2| = 2 + 2 = 4$.

If the position of the curve relative to the x-axis is disregarded and the integral is taken between the limits from 0 to 2π, the result is absurd.

Indeed, we get

$$s=\int_0^{2\pi} \sin x\,dx=\Big|_0^{2\pi} -\cos x=-\cos 2\pi+\cos 0=0.$$

2°. Formula

$$\int_a^b f(x)\,dx=F(b)-F(a)$$

holds provided that the function $y=f(x)$ is continuous on the interval $a\leqslant x\leqslant b$. For *a discontinuous function the formula may prove to be incorrect.*

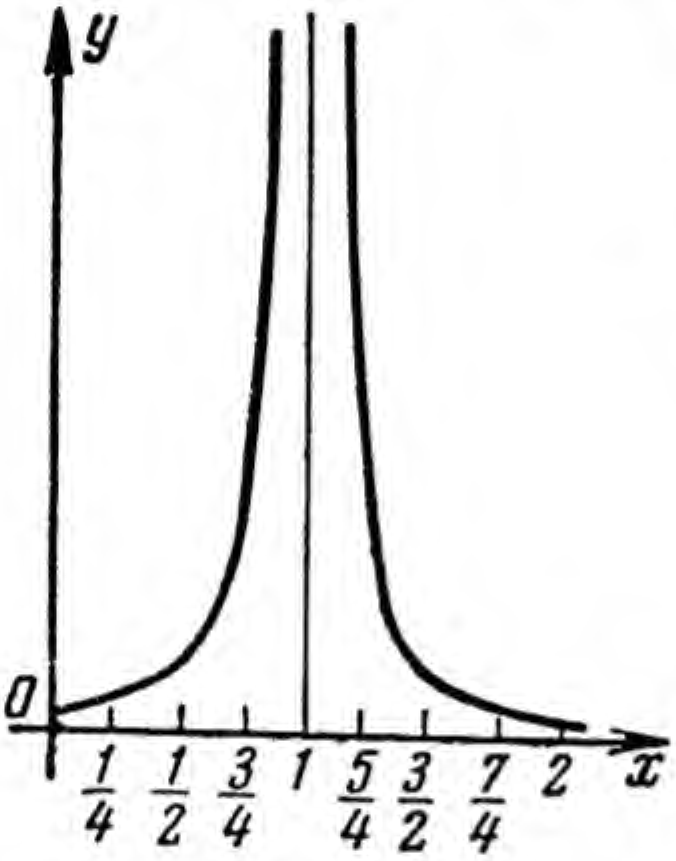

Fig. 133.

Example. Evaluate $\int_0^2 \frac{dx}{(x-1)^2}$.

Solution. Find the indefinite integral by substitution of $x-1=u$, $dx=du$:

$$\int\frac{dx}{(x-1)^2}=\int\frac{du}{u^2}=$$
$$=\frac{u^{-1}}{-1}+c=\frac{1}{1-x}+c.$$

Let us evaluate the definite integral:

$$\int_0^2\frac{dx}{(x-1)^2}=\Big|_0^2\frac{1}{1-x}=$$
$$=\frac{1}{1-2}-\frac{1}{1-0}=-1-1=-2.$$

The result is a negative number, -2, yet it is obvious that all the ordinates of the curve $y=\frac{1}{(x-1)^2}$ are positive, since the square of $x-1$ is positive for any value of x. Thus the curve lies entirely above the x-axis and the area enclosed by this curve and the x-axis cannot be expressed by a negative number. The source of the error was that the formula $\int_a^b f(x)\,dx=F(b)-F(a)$ should not have been used since on the interval $0\leqslant x\leqslant 2$ the function $y=\frac{1}{(x-1)^2}$ has a point of discontinuity at $x=1$ (Fig. 133).

Sec. 129. The Definite Integral as the Limit of a Sum

1°. Advanced courses of analysis prove a *theorem about the existence of a definite integral.* In essence it is as follows.

1. The function $y=f(x)$ is continuous on the interval $[a, b]$ (Fig. 134). Divide this interval into n subintervals (it is immaterial

whether the subintervals are equal or unequal). Denote the abscissas of the points $x_1 = a$, x_2, x_3, ..., x_k, ... x_{n+1} where $x_1 < x_2 < x_3 < \ldots < x_{n+1} = b$.

Let the lengths of the subintervals be denoted by Δx_1, Δx_2, ..., Δx_k, ..., Δx_n, where

$$\Delta x_1 = x_2 - x_1, \quad \Delta x_2 = x_3 - x_2, \ \ldots, \ \Delta x_k = x_{k+1} - x_k, \ \ldots,$$
$$\Delta x_n = x_{n+1} - x_n.$$

2. In each of the subintervals thus obtained $[x_k, x_{k+1}]$ take an arbitrary point ξ_k, $x_k \leqslant \xi_k \leqslant x_{k+1}$ and evaluate the function $f(\xi_k)$ at this point.

3. Multiply this value of the function $f(\xi_k)$ by the length of the subinterval Δx_k:

$$f(\xi_k) \cdot \Delta x_k.$$

4. Add all the products thus obtained:

$$f(\xi_1)\Delta x_1 + f(\xi_2)\Delta x_2 + \ldots + f(\xi_k)\cdot\Delta x_k + \ldots + f(\xi_n)\Delta x_n =$$
$$= \sum_a^b f(\xi)\Delta x.$$

In the symbol $\sum\limits_a^b f(\xi)\Delta x$, the letter $\sum$ signifies "sum" and $f(\xi)\Delta x$ shows that all the terms of the sum are of one and the same type: the product of $f(\xi)$ by Δx. The letters a and b denote the left and

Fig. 134.

right end points of the interval over which the operations of summation are performed. This sum, $\sum\limits_a^b f(\xi)\Delta x$, is called an *integral sum*. It is obvious that an integral sum may be constituted in an infinite variety of ways that depend solely on the manner in which the interval $[a, b]$ is subdivided into n subintervals and on the choice of the points ξ.

5. Let us consider the number of divisions, n, variable and let us assume that the interval $[a, b]$ is divided in a manner such

that for any given positive number δ (for any value of n above a certain limit) the following inequalities are satisfied:

$$\Delta x_1 < \delta, \quad \Delta x_2 < \delta, \ \ldots, \ \Delta x_k < \delta, \ \ldots, \ \Delta x_n < \delta.$$

In other words, all $\Delta x \to 0$, and the number of subintervals n becomes greater than any positive number $N = \frac{b-a}{\delta}$, i.e., $n \to \infty$. Under these conditions, each product $f(\xi_k) \cdot \Delta x_k$ is an infinitesimal since it is a product of a finite quantity $f(\xi_k)$ by an infinitesimal Δx_k. And the sum $\sum_a^b f(\xi) \Delta x$ becomes the sum of an indefinitely increasing number of infinitely small terms. In the theorem on the existence of a definite integral it is proved that this sum has a limit, which is called the definite integral $\int_a^b f(x)\,dx$, i.e.,

$$\lim_{\Delta x \to 0} \sum_a^b f(\xi) \Delta x = \int_a^b f(x)\,dx.$$

2°. Let us illustrate this theorem. AB (Fig. 135) is the graph of a continuous positive function increasing on the interval $[a, b]$. Divide the interval $[a, b]$ into n subintervals (it is immaterial whether they are equal or unequal). Let us take the left end point x_k of each of these n subintervals as the point ξ_k Then

$$f(\xi_k) = f(x_k) = y_k,$$

and the product

$$f(\xi_k) \cdot \Delta x_k = f(x_k) \cdot \Delta x_k,$$

or is equal to the area of the rectangle whose base is the segment Δx_k and whose altitude is the ordinate $f(x_k)$ of its left end point. The integral sum $\sum_a^b f(\xi) \Delta x = \sum_a^b f(x) \Delta x$ represents the area of the stepped figure $aAM_1'M_1M_2'M_2 \ldots M_nb$ which these rectangles form on the interval $[a, b]$.

The area of the curvilinear trapezoid $aABb$ is equal to the sum of the areas of the rectangles, $\sum_a^b f(x) \Delta x$, added to the sum of the areas of the curvilinear triangles (shaded portions in Fig. 135). Let σ denote the area of these triangles. Then

$$s = \sum_a^b f(x) \cdot \Delta x + \sigma.$$

The sum $\sum_a^b f(x)\,\Delta x$ of the areas of the rectangles, as Δx tends to zero, is a bounded variable quantity, because it can never be greater than the area of a rectangle with ab as its base and the

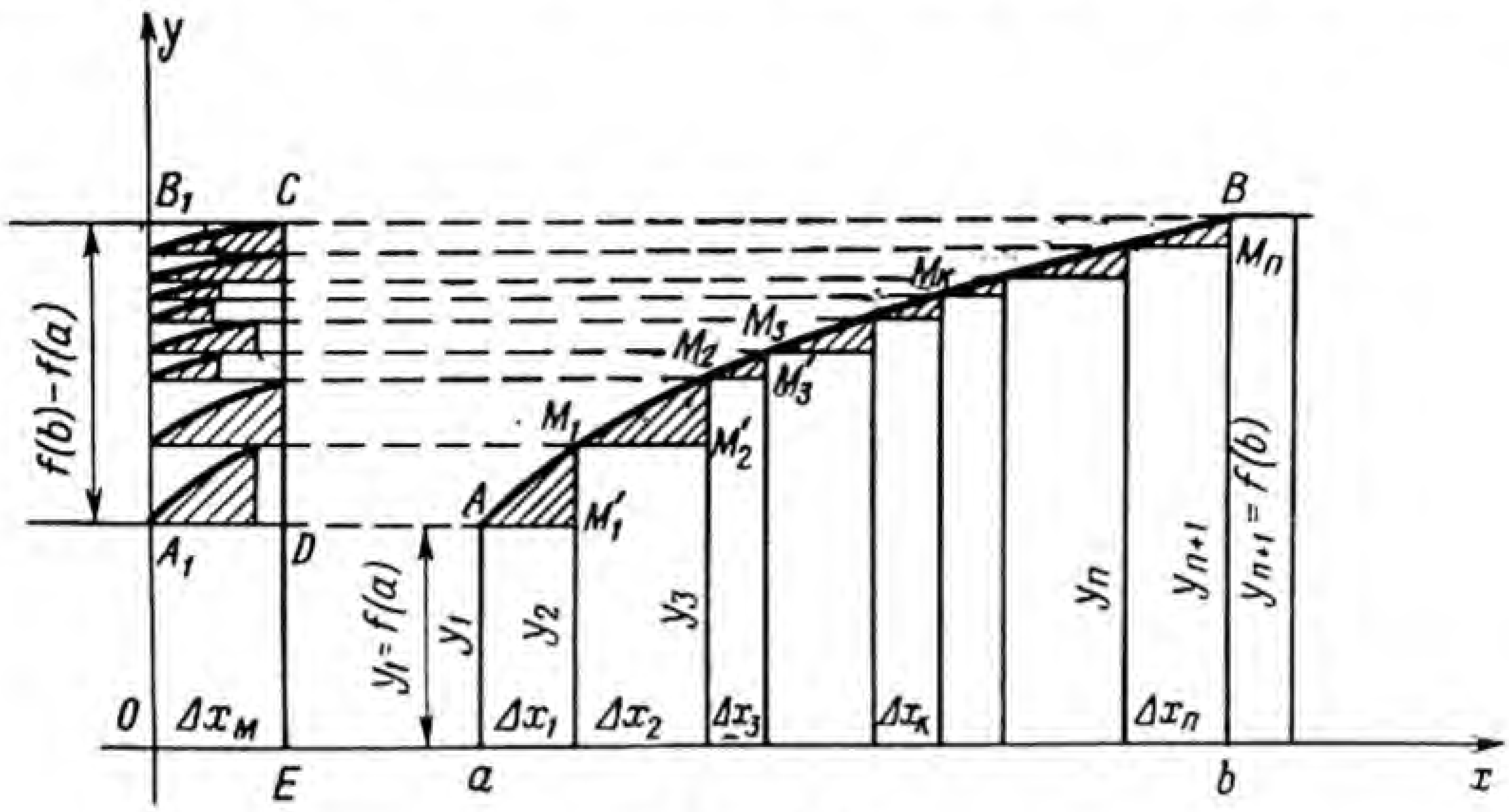

Fig. 135.

altitude equal to the largest ordinate bB. The sum is not an infinitesimal because it cannot be less than the area of a rectangle with the same base ab but altitude equal to $\frac{1}{2}(aA + bB)$.

We will show that the sum σ of the curvilinear triangles is an infinitely small quantity when $\Delta x \to 0$.

Let us take some definite number of divisions. All the segments Δx may be of different length (some of them may even be equal) and at least one of these will be greatest (in Fig. 135 there are two such largest Δx). Let us denote this largest segment by Δx_M. On the axis Ox lay off a segment from the origin O equal to Δx_M and use it as a base to construct a rectangle OB_1CE with altitude equal to bB. From the points $A, M_1, M_2, \ldots, B$ of the given curve draw perpendiculars to the y-axis and displace the curvilinear triangles $AM_1'M_1$, $M_1M_2'M_2, \ldots$ along these perpendiculars so that their vertices $A, M_1, M_2 \ldots$ lie on the y-axis. Here, the curvilinear triangles fit in the rectangle A_1B_1CD without overlapping. The base of the rectangle is Δx_M and its altitude, $f(b) - f(a)$.

Part of the area of the rectangle A_1B_1CD is not covered by the curvilinear triangles. Thus, the sum σ of the areas of the curvilinear triangles is less than the area of the rectangle A_1B_1CD, i.e.,

$$\sigma < [f(b) - f(a)] \cdot \Delta x_M.$$

This inequality holds for any value of n so long as the method of dividing the interval $[a, b]$ remains as stated in the theorem. And since by this method all the Δx (including Δx_M) become infinitely small, the product of the constant $f(b)-f(a)$ by the infinitely small quantity Δx_M is also infinitely small. If $[f(b)-f(a)]\cdot\Delta x_M < \varepsilon$, then $\sigma<\varepsilon$, i.e., σ also becomes infinitely small, as required. The equality

$$s=\sum_a^b f(x)\cdot\Delta x+\sigma$$

signifies that the constant s is equal to the sum of the variable $\sum_a^b f(x)\cdot\Delta x$ and the infinitesimal σ. Therefore (Sec. 49)

$$\lim_{\Delta x\to 0}\sum_a^b f(x)\,\Delta x=s.$$

But (Sec. 127, formula XIX)

$$s=\int_a^b f(x)\,dx.$$

Hence

$$\boxed{\lim_{\Delta x\to 0}\sum_a^b f(x)\cdot\Delta x=\int_a^b f(x)\cdot dx} \qquad \text{(XX)}$$

3°. The theorem developed in 1° is true for any kind of continuous function $f(x)$, increasing or decreasing, or increasing in some intervals of variation of the argument and decreasing in other intervals; and it is immaterial whether the function has positive or negative values.

Sec. 130. Properties of a Definite Integral

Regarding the definite integral $\int_a^b f(x)\,dx$ as the limit of a sum of infinitely small terms $f(x)\cdot\Delta x\,(\Delta x\to 0)$, with the number of terms infinitely increasing, we shall prove the properties of the definite integral.

1. *The definite integral of an algebraic sum of several functions is equal to a similar algebraic sum of the definite integrals of the functions of the terms.* Indeed,

$$\int_a^b [f(x)+\varphi(x)-\psi(x)]\,dx = \lim_{\Delta x\to 0}\sum_a^b [f(x)+\varphi(x)-\psi(x)]\,\Delta x =$$

$$=\lim_{\Delta x\to 0}\sum_a^b f(x)\,\Delta x+\lim_{\Delta x\to 0}\sum_a^b \varphi(x)\,\Delta x-\lim_{\Delta x\to 0}\sum_a^b \psi(x)\,\Delta x=$$

$$=\int_a^b f(x)\,dx+\int_a^b \varphi(x)\,dx-\int_a^b \psi(x)\,dx.$$

2. *The numerical factor of the integrand can be taken outside the integral sign.*

Proof.

$$\int_a^b c\cdot f(x)\,dx=\lim_{\Delta x\to 0}\sum_a^b c\cdot f(x)\,\Delta x=\lim_{\Delta x\to 0} c\cdot\sum_a^b f(x)\,\Delta x=$$

(since c is a common factor for all the terms of the sum)

$$=c\cdot\lim_{\Delta x\to 0}\sum_a^b f(x)\,\Delta x=c\cdot\int_a^b f(x)\,dx.$$

3. *Interchanging the limits of integration changes the sign of the definite integral.*

Proof. In the preceding section it was assumed when dividing the interval $[a, b]$ into n subintervals, that $a<b$. Now, imagine that $b<a$ and that the division is performed from the end point a towards b. Then all values of Δx will be negative. (When $b-a$ is divided, for example, into n equal subintervals, $\Delta x=\frac{b-a}{n}<0$ since $b-a<0$.)

As a consequence, the terms $f(\xi_k)\cdot\Delta x_k$ of the sum $\sum_b^a f(\xi)\,\Delta x$ and of the sum $\sum_a^b f(\xi)\,\Delta x$ are themselves sums. And their limits $\int_b^a f(x)\,dx$ and $\int_a^b f(x)\,dx$ differ in sign:

$$\int_b^a f(x)\,dx=-\int_a^b f(x)\,dx.$$

Sec. 131. Calculating Areas

1°. To find the area of the segment OAB of the parabola $y^2=2px$ (Fig. 136).

Solution. From the given equation we have

$$y=\pm\sqrt{2px}.$$

The positive values correspond to points of the curve above the x-axis, the negative values, to points below. Since the curve is symmetric about

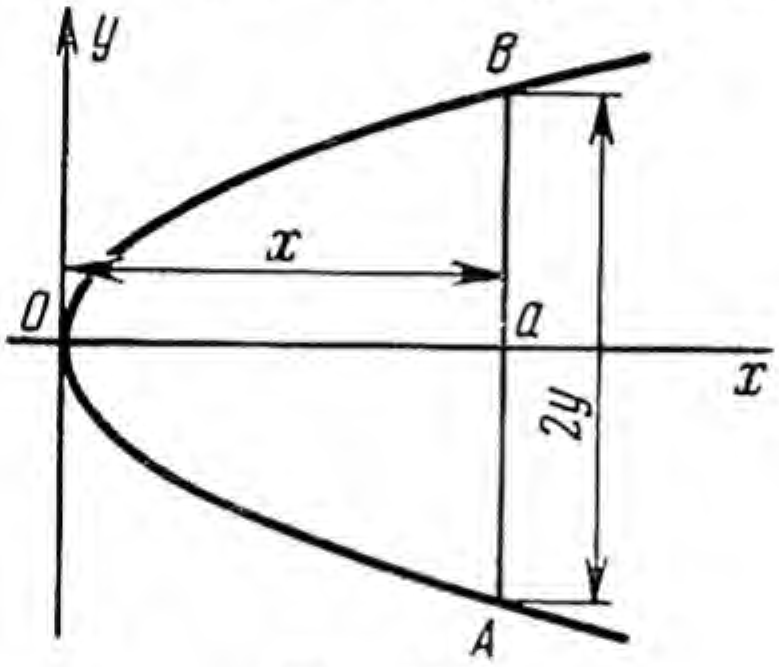

Fig. 136.

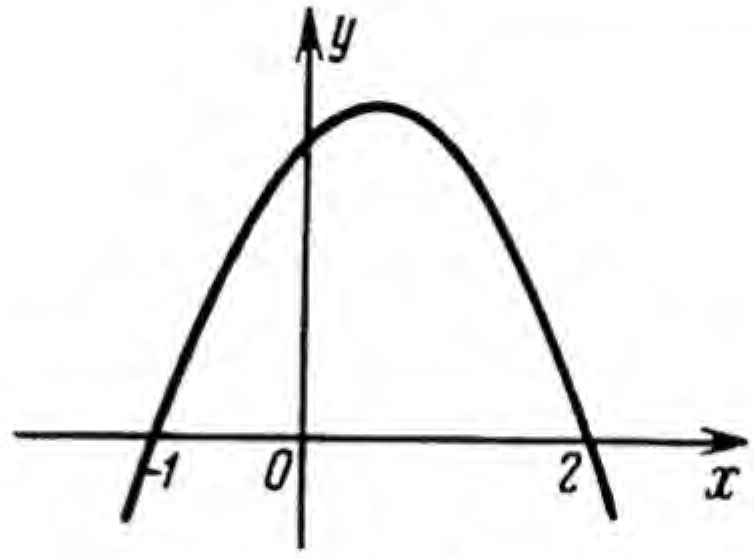

Fig. 137.

the x-axis, it is sufficient to calculate the area of the half-segment above the axis and then double the result. The argument x varies between the limits from 0 to x:

$$S_1=\int_0^x \sqrt{2px}\,dx=\sqrt{2p}\int_0^x x^{\frac{1}{2}}\,dx=$$

$$=\sqrt{2p}\,\Big|_0^x \frac{2}{3}x^{\frac{3}{2}}=\frac{2}{3}\sqrt{2p}\cdot x\cdot\sqrt{x}=\frac{2}{3}x\cdot\sqrt{2px}.$$

Since it is given that $\sqrt{2px}=y$, $S_1=\frac{2}{3}x\cdot y$.

The whole area of the segment $S=2S_1=2\cdot\frac{2}{3}x\cdot y$, or

$$S=\frac{2}{3}x\cdot 2y.$$

Here $2y$ is the length of the chord of the segment and x is its arm.

Consequently, *the area of the segment of a parabola cut off by a chord perpendicular to its axis is equal to two-thirds of the area of the rectangle constructed on this chord and its arm.*

2°. To find the area bounded by the curve $y=2+x-x^2$ and the x-axis.

Solution. An area can be bounded by two lines only if they intersect. We first find the points of intersection by solving simultaneousy the equation of the curve, $y=2+x-x^2$, and the equation of the x-axis, $y=0$; we get their points of intersection: $x_1=-1$, $x_2=2$ (Fig. 137).

The area is calculated by putting $a=-1$, $b=2$ in formula (XIX):

$$S=\int\limits_{-1}^{2}(2+x-x^2)\,dx=\Big|_{-1}^{+2}\left(2x+\frac{x^2}{2}-\frac{x^3}{3}\right)=$$

$$=\left(2\cdot 2+\frac{2^2}{2}-\frac{2^3}{3}\right)-\left(-2+\frac{1}{2}+\frac{1}{3}\right)=4\frac{1}{2}.$$

3°. To find the area enclosed between the curve $y=x^3-6x^2+11x-6$ and the ordinates $x=0$, $x=4$.

Solution. When calculating the area, one should not mechanically integrate the given expression for y within the given limits.

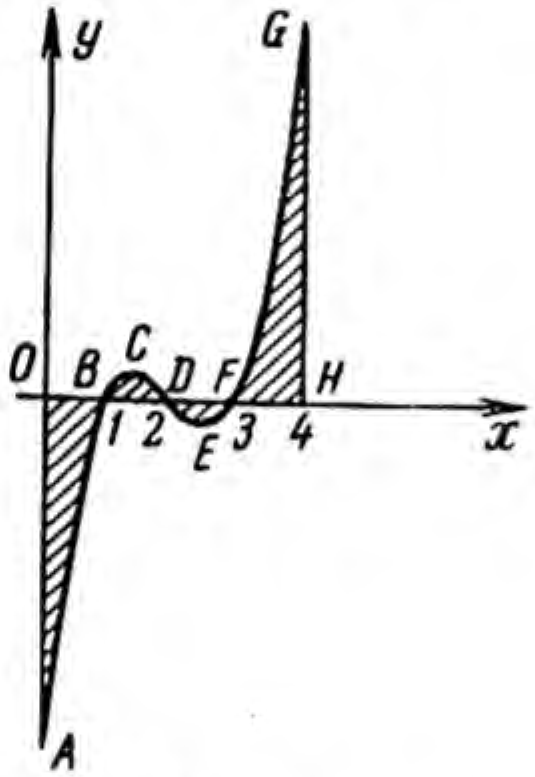

Fig. 138.

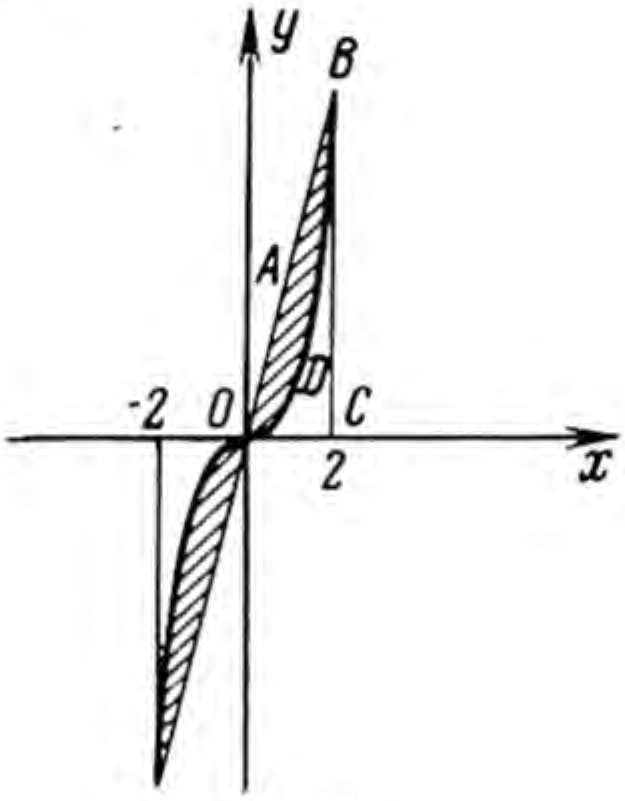

Fig. 139.

The point is that for $x=0$, $y=-6$ and for $x=4$, $y=+6$. That is, the curve lies partly below and partly above the x-axis, and, consequently, each part of the area has to be calculated separately.

To find the abscissa of the point of intersection of the curve with the x-axis, solve the equations (the equation of the x-axis is $y=0$) as a pair of simultaneous equations. We have

$$x^3-6x^2+11x-6=0.$$

This equation is solved by factorising the left-hand side:

$$x^3-6x^2+11x-6=x^3-x-6x^2+12x-6=$$

$$=x(x^2-1)-6(x^2-2x+1)=x(x-1)(x+1)-6(x-1)^2=$$

$$=(x-1)(x^2-5x+6)=0.$$

Whence

$$x-1=0 \quad \text{and} \quad x^2-5x+6=0,$$

$$x_1=1, \quad x_2=2, \quad x_3=3.$$

Thus, the curve intersects the x-axis at three points whose abscissas are, respectively, equal to 1, 2, 3 (Fig. 138). To calculate the required area, find the areas OAB, BCD, DEF, FGH and then add them together. Further, noting that the curve is symmetric about the point (2, 0), we conclude that it is sufficient to calculate the first two areas or the last two areas and

then double their sum.

$$\text{Area } OAB=\int_0^1 (x^3-6x^2+11x-6)\,dx=\Big|_0^1 \left(\frac{x^4}{4}-2x^3+\frac{11}{2}x^2-6x\right)=$$
$$=\frac{1}{4}-2+\frac{11}{2}-6=-\frac{9}{4}.$$

The minus sign merely indicates that the area OAB lies below the x-axis. Its area is equal to the absolute value of the result obtained:

$$\text{area } OAB=\frac{9}{4} \text{ square units.}$$

$$\text{Area } BCD=\int_1^2 (x^3-6x^2+11x-6)\,dx=\Big|_1^2 \left(\frac{x^4}{4}-2x^3+\frac{11}{2}x^2-6x\right)=$$
$$=(4-16+22-12)+\frac{9}{4}=\frac{1}{4}.$$

Area $BCD=\frac{1}{4}$ square units.

The required area

$$s=2\,(\text{ar. } OAB+\text{ar. } BCD)=2\left(\frac{9}{4}+\frac{1}{4}\right)=5 \text{ square units.}$$

4°. Calculate the area enclosed between the curves $y=x^3$ and $y=4x$.

Solution. First determine the boundaries of the area. To do this, solve the equations simultaneously:

$$x^3-4x=0,\quad x(x^2-4)=0,\quad x_1=-2,\quad x_2=0 \text{ and } x_3=2.$$

Sketch the parabola from the points:

x	-2	-1	$-\frac{1}{2}$	0	$+\frac{1}{2}$	$+1$	$+2$
y	-8	-1	$-\frac{1}{8}$	0	$+\frac{1}{8}$	$+1$	$+8$

and connect the points $(-2, -8)$ and $(2, 8)$ by a straight line (Fig. 139).

It is required to find the area of the shaded portion. It consists of two equal parts. Determine the area $OABD$ as the difference between the areas of the rectilinear triangle $OABC$ and the curvilinear triangle $ODBC$.

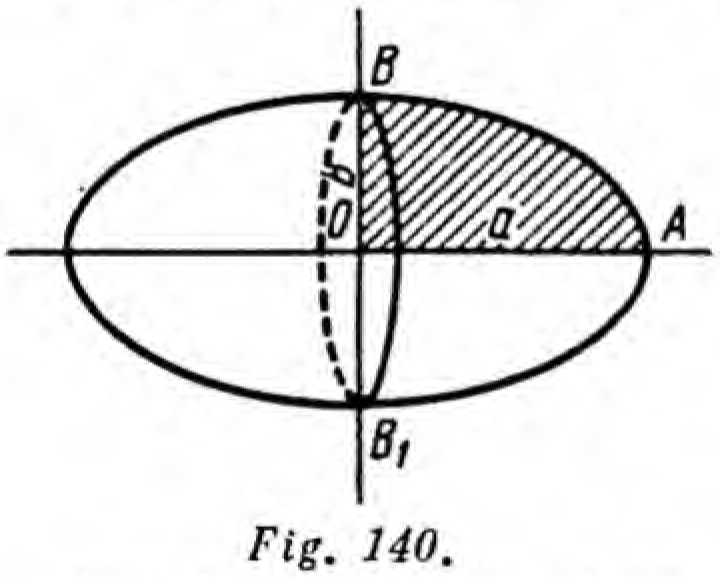

Fig. 140.

$$\text{Area } OABC=\int_0^2 4x\,dx=\Big|_0^2 2x^2=8.$$

$$\text{Area } ODBC=\int_0^2 x^3\,dx=\Big|_0^2 \frac{x^4}{4}=4.$$

Area $OABD=8-4=4$. Hence the required area is 8 square units.

5°. Find the area of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Solution. Since the ellipse is symmetric about the x-axis and y-axis, its area $s=4\cdot$area OAB (Fig. 140).

From the equation of the ellipse,

$$y=\frac{b}{a}\sqrt{a^2-x^2}.$$

Only the positive root is considered since the values of y in the portion of the ellipse under consideration (first quadrant) are positive.

$$\text{Area } OAB=\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx=\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx.$$

By formula (XVII):

$$\int \sqrt{a^2-x^2}\,dx=\frac{1}{2}a^2\arcsin\frac{x}{a}+\frac{1}{2}x\sqrt{a^2-x^2}+c.$$

$$\text{Area } OAB=\frac{b}{a}\Big|_0^a\left(\frac{1}{2}a^2\arcsin\frac{x}{a}+\frac{1}{2}x\sqrt{a^2-x^2}\right)=$$

$$=\frac{b}{2a}(a^2\arcsin 1+a\cdot 0-a^2\cdot\arcsin 0-0\cdot a)=$$

$$=\frac{b}{2a}\left(a^2\frac{\pi}{2}-a^2\cdot 0\right)=\frac{b}{2a}\cdot a^2\cdot\frac{\pi}{2}=\frac{\pi ab}{4}.$$

The total area of the ellipse $S=4\cdot\frac{\pi ab}{4}=\pi ab.$

Note that the formula for the area of a circle, $s=\pi r^2$, is a particular case of the formula for the area of an ellipse when $a=b=r$.

Sec. 132. Volume of a Pyramid

To illustrate the application of a definite integral as the limit of a sum, let us perform the operations given in 1° of Sec. 129 to find the volume v of a triangular pyramid of base area S square units and altitude h linear units.

Let us divide the altitude $OM=h$ (Fig. 141) of the pyramid into n segments (it is immaterial whether the segments are equal or unequal): $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$. Through the points of division of the altitude h draw planes parallel to the base of the pyramid. The area of the polygon of cross-section s thus obtained has the same relation to the base of the pyramid S as the squares of their distances from the vertex of the pyramid.

Denoting the distance from the vertex to the cross-section under consideration abc by x, we get

$$\frac{s}{S}=\frac{x^2}{h^2};\qquad s=\frac{S}{h^2}x^2.$$

Thus, the area s is a continuous function of x defined on the interval $[0, h]$:

$$s = f(x) = \frac{S}{h^2} x^2.$$

Let us construct a prism of altitude Δx on each polygon cross-section in such a manner that the side edges of each prism are parallel to the edge OA, and the polygon cross-section is the upper base of the prism. The aggregate of these prisms constitutes a stepped structure entirely contained in the pyramid.

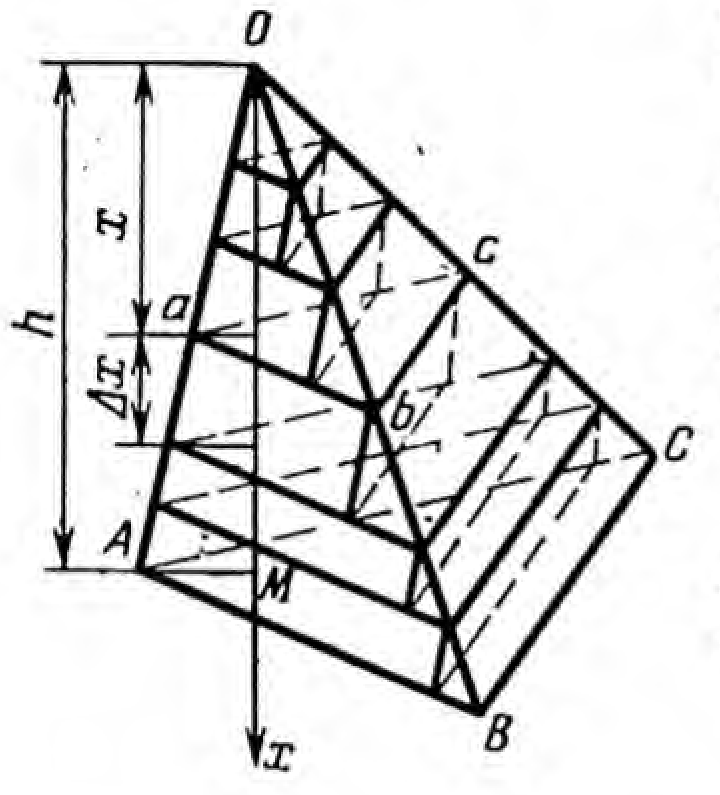

Fig. 141.

The volume of each prism is equal to the product of the area of the cross-section $f(x)$ by the altitude of the prism Δx:

$$f(x) \cdot \Delta x = \frac{S}{h^2} x^2 \cdot \Delta x.$$

The volume of the stepped structure consisting of the constructed prisms is the integral sum

$$\sum_0^h f(x) \cdot \Delta x = \sum_0^h \frac{S}{h^2} \cdot x^2 \Delta x.$$

The volume of the pyramid is the limit of this integral sum as $\Delta x \to 0$:

$$v = \lim_{\Delta x \to 0} \sum_0^h \frac{S}{h^2} x^2 \Delta x = \int_0^h \frac{S}{h^2} x^2 \, dx = \frac{s}{h^2} \int_0^h x^2 \, dx =$$

$$= \frac{S}{h^2} \left. \frac{x^3}{3} \right|_0^h = \frac{S}{h^2} \cdot \frac{h^3}{3} = \frac{1}{3} S \cdot h.$$

Thus, *the volume of a (triangular) pyramid is equal to one-third the product of the area of the base by the altitude.*

Sec. 133. Volume of a Solid of Revolution

1°. In Fig. 142, the figure $aABb$ is formed by the arc AB of a continuous curve $y = f(x)$, by the ordinates of the points A and B, and by the segment of the x-axis cut out by these ordinates. By revolving the figure $aABb$ about the x-axis we obtain the surface of revolution ABB_1A_1; the body enclosed within this surface between the circles AA_1 and BB_1 is called a solid of revolution. Let us calculate its volume. Construct a system of rectangles for the area of $aABb$ (Sec. 129, 2°). When revolved about the x-axis, the polygons generate cylinders. The aggregate of these cylinders represents a stepped body in the shape of a stepped pulley. The volume of each cylinder is equal to $\pi y^2 \Delta x$ and the

volume of the stepped body is

$$\sum_a^b \pi y^2 \Delta x.$$

Here πy^2 is a continuous function of x, since it is given that $y=f(x)$ is continuous on the interval $[a, b]$ (Sec. 72, 4°), and,

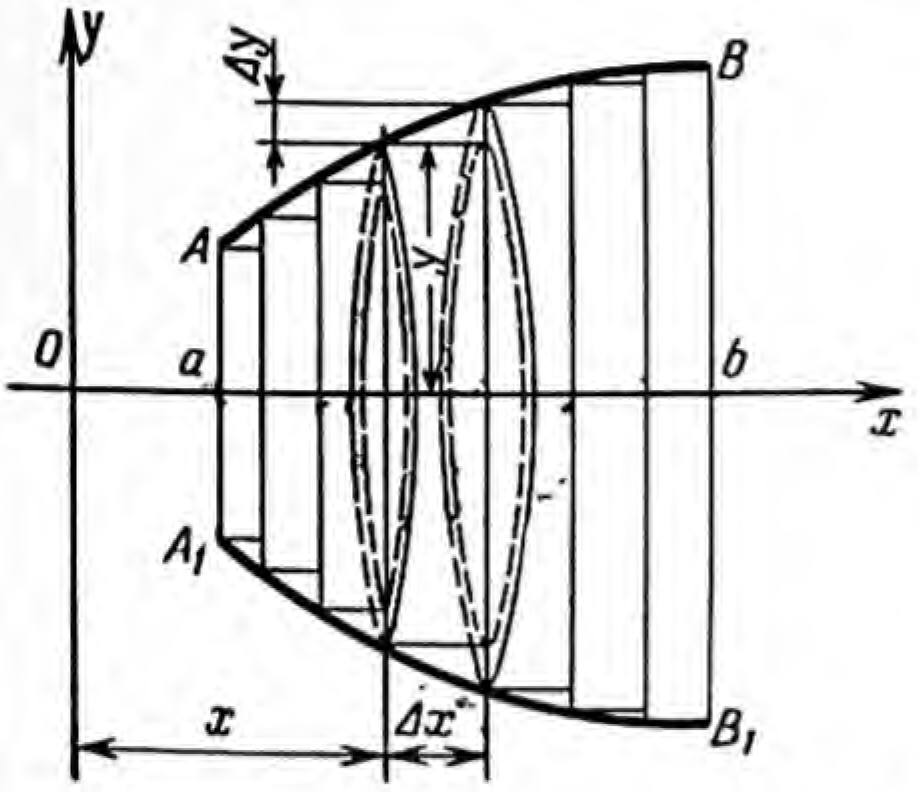

Fig. 142.

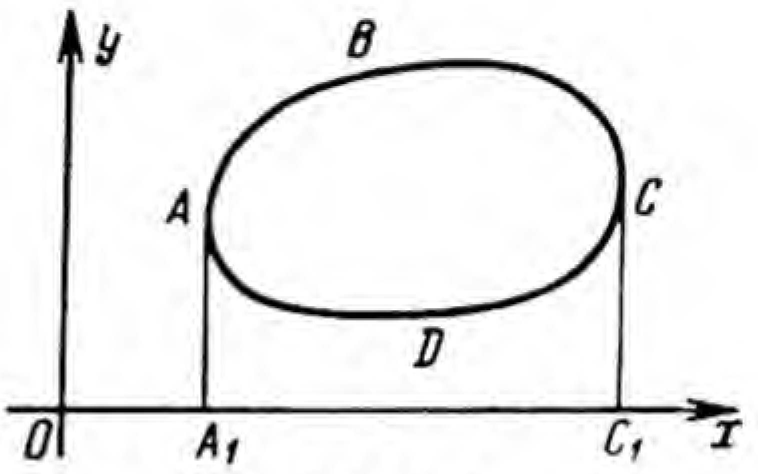

Fig. 143.

consequently, the integral sum $\sum_a^b \pi y^2 \Delta x$ has a limit when $\Delta x \to 0$ (Sec. 129, 1°). This limit is the volume v of the solid of revolution:

$$v = \lim_{\Delta x \to 0} \sum_a^b \pi y^2 \Delta x = \int_a^b \pi y^2\, dx = \pi \int_a^b y^2\, dx.$$

Hence, the volume of a solid of revolution is

$$\boxed{v = \pi \int_a^b y^2\, dx, \quad \text{or} \quad v = \pi \int_a^b f^2(x)\, dx} \qquad \text{(XXI)}$$

2°. If the solid is generated by a closed curve, as, for example, $ABCD$ in Fig. 143, then the required volume is equal to the difference between the volumes generated by the revolution of the curvilinear trapezoids A_1ABCC_1 and A_1ADCC_1.

Let y_1 denote the current coordinate of the upper part of the curve (ABC) and let y_2 denote the current coordinate of the lower part (ADC). We get

$$\boxed{v = \pi \int_a^b (y_1^2 - y_2^2)\, dx} \qquad \text{(XXII)}$$

Sec. 134. Calculating the Volumes of Solids of Revolution

1°. The volume of a right circular cone.

A right circular cone of altitude h and radius r is generated by revolving the right triangle OAB (Fig. 144), the legs of which are h and r, about the leg OA equal to h.

Let O be the origin and let OA be the axis Ox. Then the equation of the straight line OB is $y=\frac{r}{h}x$.

The volume of the cone (formula XXI) is

$$v=\pi\int_0^h y^2\,dx=\pi\int_0^h \frac{r^2}{h^2}x^2\,dx=\frac{\pi r^2}{h^2}\Big|_0^h \frac{x^3}{3}=\frac{\pi r^2}{h^2}\cdot\frac{h^3}{3}=\frac{1}{3}\pi r^2\cdot h.$$

Since πr^2 is the area of the base, *the volume of the cone is equal to one-third the product of the area of the base and the altitude.*

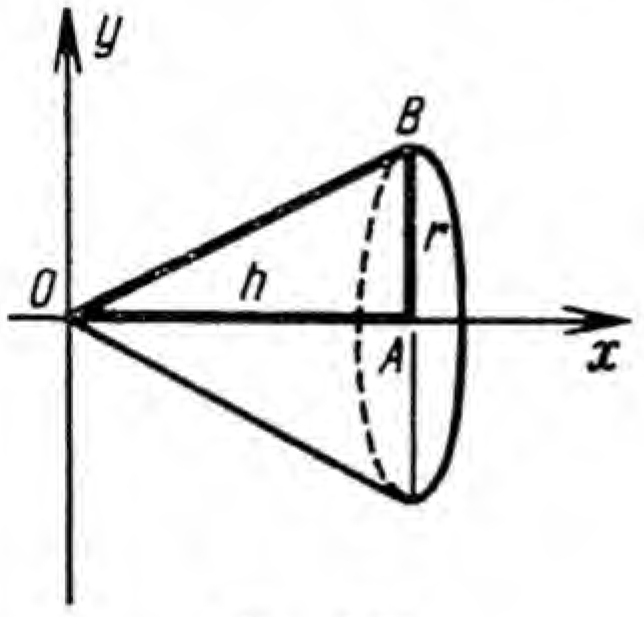

Fig. 144.

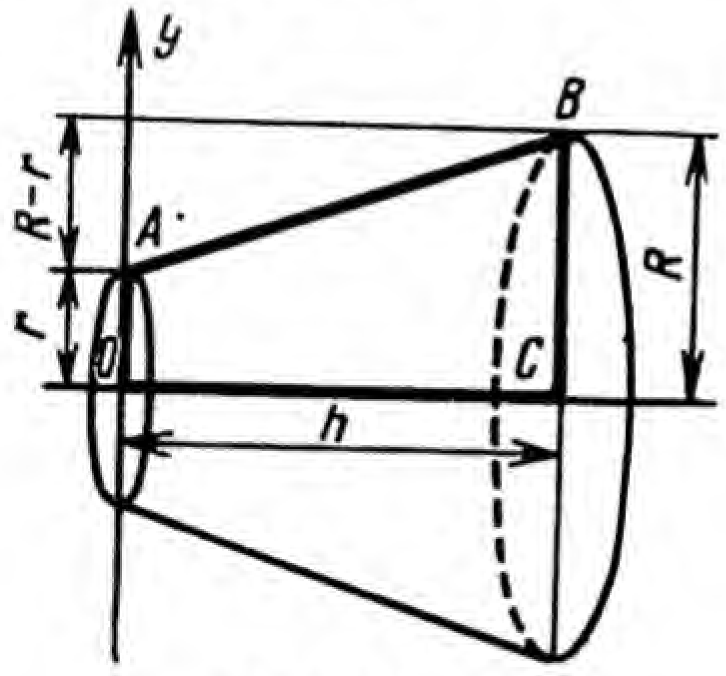

Fig. 145.

2°. The volume of a truncated cone.

A truncated cone of altitude h and bases of radii r and R is obtained by revolving the rectangular trapezoid $OABC$ (Fig. 145) about OC as the axis. Arranging the coordinate axes as shown in Fig. 145, write the equation of the straight line AB:

$$y=\frac{R-r}{h}x+r.$$

The volume of a truncated cone (formula XXI) is

$$v=\pi\int_0^h\left(\frac{R-r}{h}x+r\right)^2dx.$$

Assuming $\frac{R-r}{h}x+r=u$, we have $dx=\frac{h}{R-r}\,du$,

$$\int\left(\frac{R-r}{h}x+r\right)^2dx=\frac{h}{R-r}\int u^2\,du=\frac{h}{R-r}\cdot\frac{u^3}{3}+c=$$

$$=\frac{h}{3(R-r)}\cdot\left(\frac{R-r}{h}x+r\right)^3+c,$$

$$v=\frac{\pi h}{3(R-r)}\Big|_0^h\left(\frac{R-r}{h}x+r\right)^3=\frac{\pi h}{3(R-r)}\left[\left(\frac{R-r}{h}\cdot h+r\right)^3-r^3\right]=$$

$$=\frac{\pi h(R^3-r^3)}{3(R-r)}=\frac{1}{3}\pi h(R^2+Rr+r^2).$$

Removing brackets, we get the volume in the following form:

$$v=\frac{1}{3}\pi R^2h+\frac{1}{3}\pi r^2h+\frac{1}{3}\pi Rrh.$$

Thus, *the volume of a truncated cone is equal to the sum of the volumes of three cones whose altitudes are the same as that of the truncated cone while the bases are the lower base of the truncated cone (first cone), the upper base of the truncated cone (second cone), and a circle of area equal to the geometrical mean of the areas of the two bases of the truncated cone (third cone).*

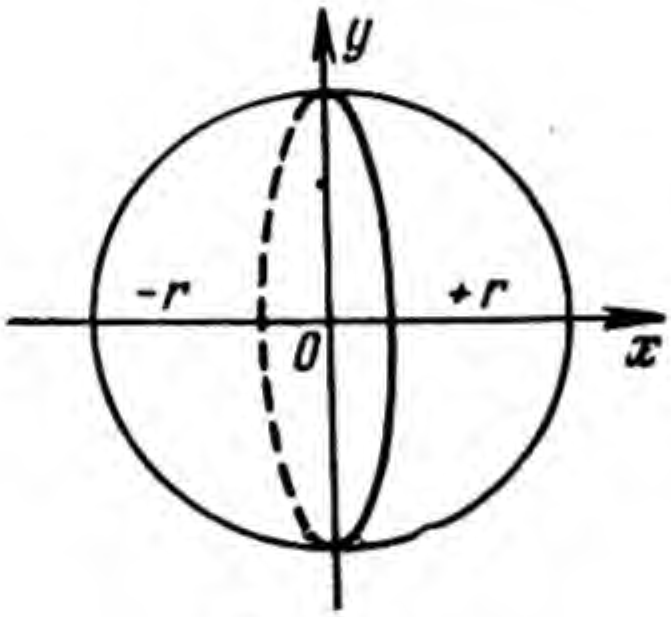

Fig. 146.

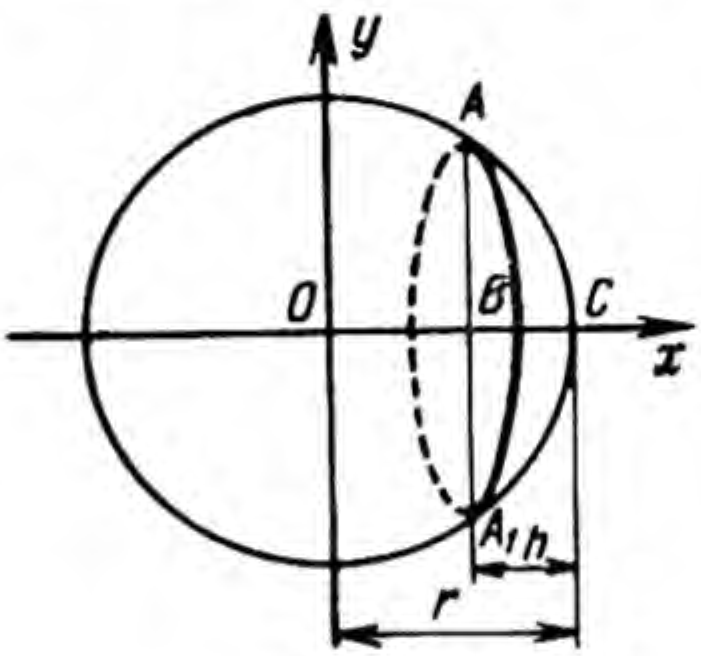

Fig. 147.

3°. **The volume of a sphere.** A sphere of radius r is generated by revolving a circle ($x^2+y^2=r^2$) about the axis Ox (Fig. 146). Hence (formula XXI):

$$v=\pi\int_{-r}^{+r}y^2\,dx=2\pi\int_0^{+r}(r^2-x^2)\,dx=2\pi\Big|_0^r\left(r^2x-\frac{x^3}{3}\right)=$$

$$=2\pi\left(r^3-\frac{r^3}{3}\right)=\frac{4}{3}\pi r^3.$$

Representing $\frac{4}{3}\pi r^3$ in the form $4\pi r^2\cdot\frac{1}{3}r$, we get: *the volume of a sphere is equal to the product of four times the area of a great circle of the sphere by one-third the radius.*

4°. **The volume of a spherical segment.** The portion ACA_1 (in Fig. 147) of a sphere cut off by means of a plane AA_1 is called a spherical segment. It is evident that a spherical segment may be considered a solid of revolution generated by revolving a segment, ABA_1C, of a circle about its altitude BC. We consider the radius of the circle, r, and the altitude, h, of the segment known.

The volume of a spherical segment is

$$v=\pi\int_{r-h}^{r} y^2\,dx=\pi\int_{r-h}^{r}(r^2-x^2)\,dx=\pi\Big|_{r-h}^{r}\left(r^2x-\frac{x^3}{3}\right)=$$

$$=\pi\left\{\left(r^3-\frac{r^3}{3}\right)-\left[r^2(r-h)-\frac{(r-h)^3}{3}\right]\right\}=\pi h^2\left(r-\frac{1}{3}h\right).$$

πh^2 may be considered the area of a circle of radius h, and $r-\frac{1}{3}h$ the altitude. Then the *volume of a spherical segment is equal to the volume of a cylinder whose base radius is the altitude of the segment and the altitude of the cylinder is equal to the radius of the sphere minus a third of the altitude of the segment.*

Sec. 135. Pressure of Liquids

1°. As illustrations of evaluating quantities as the limiting value of a sum of infinitesimals, let us determine the pressure of a liquid and the work done by a force.

The pressure exerted by a liquid on s square units of a horizontal area is, as we know, equal to the weight of a column of liquid having this area for its base and the depth of the area (its distance from the free surface of the luquid) for its altitude. Denoting the specific gravity of the liquid by γ and the altitude of the column by h, we find that the pressure a liquid exerts on an area s square units is

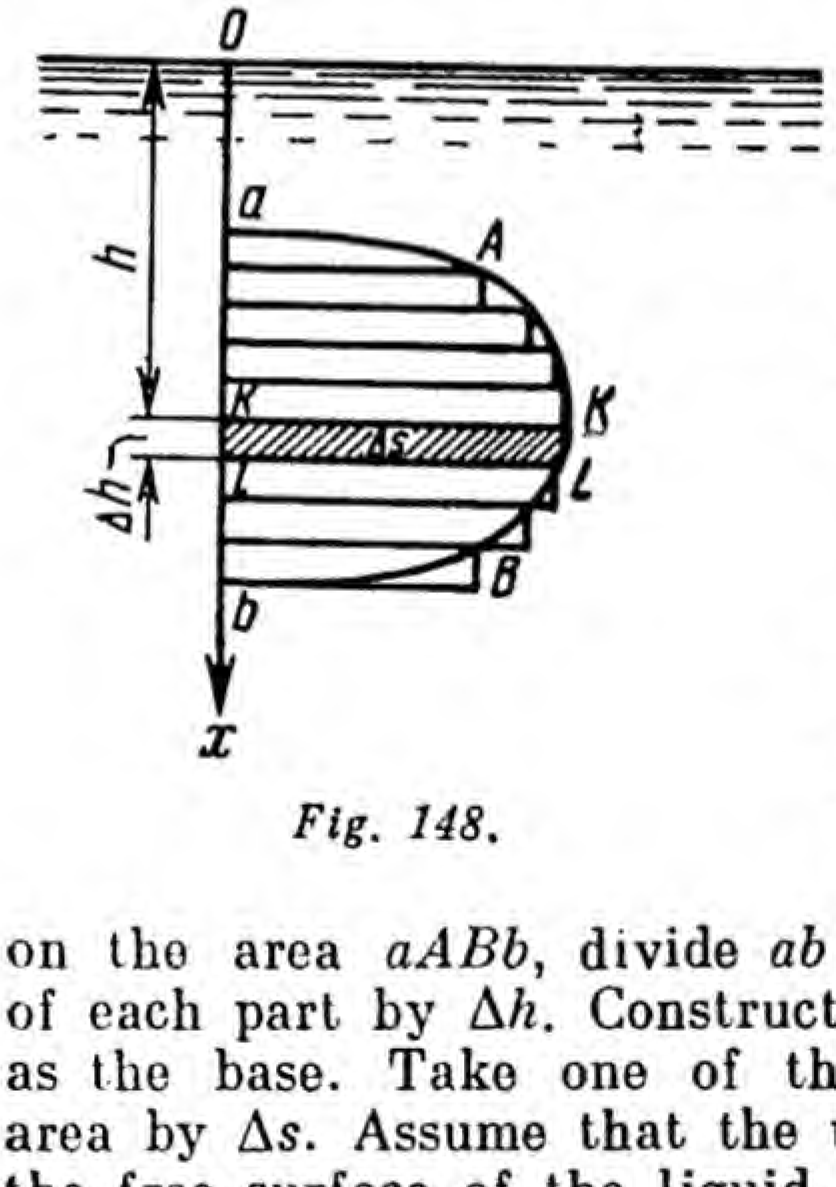

Fig. 148.

$$P=\gamma\cdot h\cdot s.$$

For any given surface s, the pressure of the liquid is a function of the depth h. Let $aABb$ (Fig. 148) be part of a vertical wall, for example, of a pool filled with liquid. To determine the pressure exerted on the area $aABb$, divide ab into n parts and denote the length of each part by Δh. Construct a rectangle on each part Δh taken as the base. Take one of these rectangles, $kKLl$ and denote its area by Δs. Assume that the upper side kK is at a depth h from the free surface of the liquid. Imagine that this rectangle $kKLl$ is situated at the depth h not in a vertical position but horizontally. Then it experiences a pressure equal to

$$\gamma h\Delta s.$$

We know that the pressure of a liquid at each point is equal in all directions. Therefore, if the rectangle $kKLl$ is now regarded as

vertical, the pressure on it will be somewhat more than in the horizontal position because the pressure on its lower border lL is somewhat greater than that on its upper border kK.

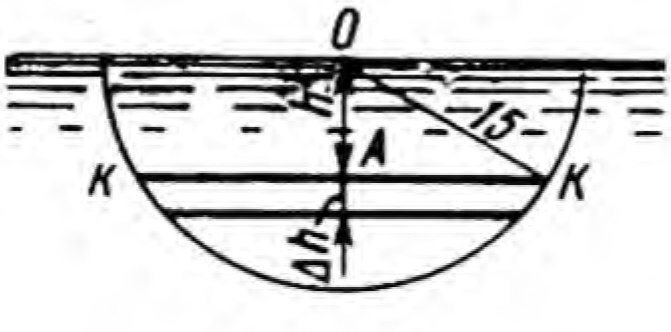

Fig. 149.

Let us assume that the pressure over the entire altitude Δh is the same as that in the upper border kK. Then the pressure on the rectangle is equal to

$$\gamma \cdot h\Delta s.$$

The area of the rectangle $kKLl$ $\Delta s = kK \cdot \Delta h$. The value of kK varies with the depth h of immersion kK, i.e., kK is a function of h:

$$kK = f(h).$$

It is evident that kK is a function continuous on the interval $[a, b]$. As a result, in the product

$$\gamma \cdot h \cdot \Delta s = \gamma \cdot h \cdot f(h) \cdot \Delta h$$

the function $\gamma \cdot h \cdot f(h)$ is continuous, and the integral sum

$$\sum_a^b \gamma h \Delta s$$

has a limit when $\Delta h \longrightarrow 0$. This limit represents the pressure of the liquid on the entire area s.

Hence, the pressure on the whole area of the wall $aABb$ is

$$\boxed{P = \lim_{\Delta h \to 0} \sum_a^b \gamma \cdot h \cdot \Delta s = \gamma \int_a^b h\, ds} \qquad \text{(XXIII)}$$

Here, *the depth h serves as the independent variable, and, prior to integration, the area of the rectangle ds must be expressed in terms of h and Δh; the upper and lower boundaries of the area over which the pressure is determined serve as the limits of integration.*

2°. **Example.** A porthole of diameter 30 cm in the vertical side of a ship is half immersed in water. Find the pressure exerted on the immersed part (Fig. 149).

Solution: The pressure is given by the formula (XXIII):

$$P=\gamma\int_a^b h\,ds.$$

$$\text{Area } ds=kK\cdot dh.$$

We find from the triangle AOK that $kK=2\sqrt{15^2-h^2}=$ $=2\sqrt{225-h^2}$, and, consequently, $ds=2\sqrt{225-h^2}\cdot dh$.

For water, $\gamma=1$ g. The limits of the integration are $a=0$, $b=15$. Hence

$$P=2\int_0^{15} h\sqrt{125-h^2}\cdot dh.$$

The indefinite integral $\int h\cdot\sqrt{225-h^2}\,dh$ is found by substitution of $225-h^2=u^2$, $-2h\,dh=2u\,du$, $h\,dh=-u\,du$:

$$2\int_0^{15} h\sqrt{225-h^2}\cdot dh=-\frac{2}{3}\Big|_0^{15}\left(\sqrt{225-h^2}\right)^3.$$

$$P=\frac{2}{3}\cdot 15^3=2250\text{ g}.$$

Sec. 136. Work Done by a Force

1°. If the force is constant its numerical value is the same at all points of the path; but if the force is variable, then at different points of the path its numerical value is, generally speaking, different, and to every value of path traversed there corresponds a definite value of the acting force F. Therefore, the force F may be regarded as a function of the distance s:

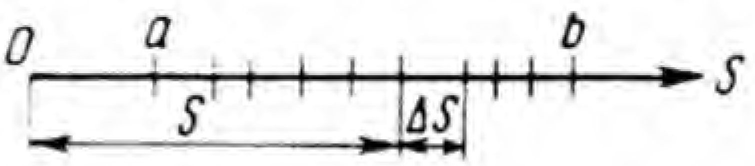

Fig. 150.

$$F=f(s).$$

We shall consider only continuous variation of the force, i.e., we take $f(s)$ as a continuous function of s and assume that the direction of force coincides with the direction of motion.

To determine the amount of work done by the variable force F on a rectilinear portion ab of the path (Fig. 150), we divide $[a, b]$ into a large number of subintervals n, and denote the length of each subinterval by Δs.

The value of force at the commencement of any particular subinterval Δs is equal to $f(s)$, where s is the distance covered up to the beginning of the subinterval under study.

Let us assume that the force $f(s)$ changes its value only at the beginning of each subinterval Δs and that it remains constant throughout Δs. Then the work done by this constant force $f(s)$ over the subinterval Δs will be expressed by the product $f(s)\cdot\Delta s$. This we know from physics. Of course, this product (which is

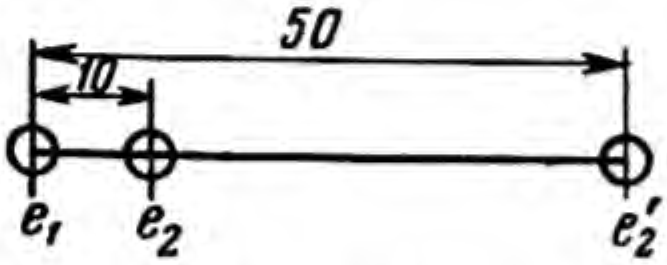

Fig. 151.

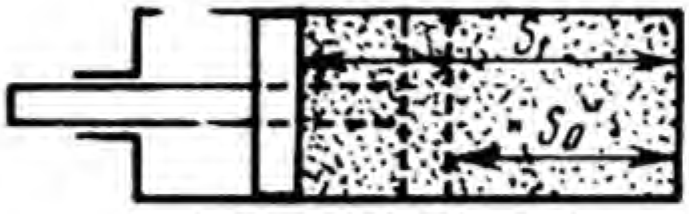

Fig. 152.

generally called the "elementary work") differs from the actual work done over the subinterval Δs, and the sum $\sum\limits_a^b f(s)\cdot\Delta s$ differs from the actual work done on $[a, b]$.

Since $f(s)$ is a continuous function on the interval $[a, b]$, the integral sum $\sum\limits_a^b f(s)\cdot\Delta s$ has a limit when Δs tends to zero:

$$\lim_{\Delta s\to 0}\sum_a^b f(s)\cdot\Delta s,$$

equal to the work w done by the force $f(s)$ on the path ab.

Hence,

$$\boxed{w=\lim_{\Delta s\to 0}\sum_a^b f(s)\cdot\Delta s=\int_a^b f(s)\,ds} \qquad \text{(XXIV)}$$

2°. **Example 1.** Two electric charges, $e_1 = +100$ and $e_2 = +50$ electrostatic units, are fixed at a separation of 10 cm. Find the amount of work done by the force of repulsion of the charges if e_2 is released and recedes to 50 cm from e_1 (Fig. 151).

Solution. The force of repulsion F acting on the moving charge e_2 is given by Coulomb's law:

$$F=\frac{e_1\cdot e_2}{r^2}=\frac{100\cdot 50}{r^2}=\frac{5000}{r^2},$$

where r is the distance in centimetres between the charges. The force F does not remain constant: it continually decreases as the charge e_2 recedes. In other words, it is indeed a function of the distance travelled, as in the general case just examined.

The limits of variation of the independent variable r are: the initial distance between the charges (10 cm) and the final distance (50 cm). By formula (XXIV), the work done by the force of repulsion is

$$w = \int_{10}^{50} \frac{5000}{r^2}\, dr = -5000\left(\frac{1}{50} - \frac{1}{10}\right) = 400 \text{ ergs.}$$

3°. **Example 2.** Gas is enclosed in a cylinder with a movable piston, the cross-sectional area of which is a square units. Assuming that the law of Boyle-Mariotte, $p \cdot v = k$, holds for the expanding gas, calculate the work done by the pressure of the gas as it increases in volume from v_0 to v_1 (Fig. 152).

Solution. Let v be the volume of the gas in the cylinder, and p the pressure on unit area of the piston. Since the area of the piston is a units, the pressure exerted by the gas on the piston is $p \cdot a$. Let us assume that when the volume of the gas increases from v_0 to v_1, the piston travels a distance $s = s_1 - s_0$. Then the work done by the pressure of the gas is given by formula (XXIV):

$$w = \int_{s_0}^{s_1} p \cdot a \cdot ds.$$

Note that here the pressure p is a variable quantity dependent on v, or $p = \frac{k}{v}$. Let ds be expressed as a function of v. Assume that the piston moves by ds when the volume of the gas increases by dv. Then

$$dv = a \cdot ds \text{ and } ds = \frac{dv}{a}.$$

Substituting into the integral the calculated values of p and ds and replacing the limits of variation of s by the corresponding limits of variation of v, we get

$$w = \int_{s_0}^{s_1} p \cdot a \cdot ds = \int_{v_0}^{v_1} \frac{k}{v}\, a \cdot \frac{dv}{a} = k \int_{v_0}^{v_1} \frac{dv}{v} = k \ln \frac{v_1}{v_0}.$$

D. SUPPLEMENT

CHAPTER XII

DIFFERENTIATION OF FUNCTIONS OF SEVERAL VARIABLES

Sec. 137. Partial Derivatives and Partial Differentials. Total Differential and Its Application

1°. If a quantity is dependent not on one independent variable but on two or more independent variables, it is called a function of two, or many, independent variables. For example, a volume of gas is a function of two variables: temperature and pressure; the amount of heat generated in a conductor is a function of three independent variables: current, resistance of the conductor, and time.

Let u be a function of several variables, say, three:

$$u = f(x, y, z).$$

Let one of these independent variables, say x, have an increment Δx and let us hold the other independent variables constant. Then u will receive a new value:

$$u + \Delta_x u = f(x + \Delta x, y, z).$$

Subtracting the initial value of the function from the second value, we obtain the partial increment of the function u with respect to x:

$$\Delta_x u = f(x + \Delta x, y, z) - f(x, y, z).$$

The limit of the ratio of the partial increment of the function u with respect to x to the increment Δx as $\Delta x \longrightarrow 0$ is called the *partial derivative of u with respect to x* and is denoted by $\frac{\partial u}{\partial x}$ or u_x

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}.$$

In an analogous manner, y can be treated as the independent variable and x and z as constants. Then, the *partial derivative of*

u with respect to y is

$$\frac{\partial u}{\partial y}=\lim_{\Delta y \to 0}\frac{f(x,\ y+\Delta y,\ z)-f(x,\ y,\ z)}{\Delta y};$$

and the partial derivative of u with respect to z is

$$\frac{\partial u}{\partial z}=\lim_{\Delta z \to 0}\frac{f(x,\ y,\ z+\Delta z)-f(x,\ y,\ z)}{\Delta z}.$$

2°. Partial derivatives are found by the formulas and rules of differentiation of a function of one argument (Ch. VII), since when evaluating a partial derivative with respect to x, u is regarded as the function of a single argument x, when with respect to y, as a function of a single argument y, etc.

Example. $u=x^2y^3z^4$. Find the partial derivatives with respect to x, y, and z.

Solution. $\frac{\partial u}{\partial x}=2xy^3z^4$ (x is the variable; y and z are constants);
$\frac{\partial u}{\partial y}=3x^2y^2z^4$ (y is the variable; x and z are constants);
$\frac{\partial u}{\partial z}=4x^2y^3z^3$ (z is the variable; x and y are constants).

Note that $\frac{\partial u}{\partial x}$ is not meant to represent a ratio. It is a single symbol.

3°. The product of the partial derivative of a function with respect to some variable, for instance x, by the differential dx of this variable is called the *partial differential of the function* with respect to this variable. The partial differential of the function u with respect to the variable x is denoted by the symbol d_xu. By definition,

$$d_xu=\frac{\partial u}{\partial x}\cdot dx$$

$$d_yu=\frac{\partial u}{\partial y}\cdot dy;$$

etc.

4°. In applications of analysis it is frequently necessary to determine the increment of a function due to increments in all of its independent variables. The increment of the function u is different from the sum of its partial increments with respect to all its arguments. For example, the function $u=xy$ has partial increments Δ_xu and Δ_yu (cross-hatching in Fig. 153). The full increment Δu differs from the sum $\Delta_xu+\Delta_yu$ by the quantity $\Delta x\cdot\Delta y$.

For an approximate calculation of the increment of the function $u=f(x,\ y)$, the following formula is used:

$$\Delta u\approx\frac{\partial u}{\partial x}\cdot dx+\frac{\partial u}{\partial y}\cdot dy.$$

In advanced courses of analysis it is shown that if the function $u=f(x, y)$ defined for $a<x<b$ and $c<y<d$ has partial derivatives with respect to x and y, and these partial derivatives are functions continuous at the point (x, y), then the increment Δu differs from the sum $\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$ by an infinitesimal of higher order than $|\Delta x|+|\Delta y|$.

Given the above conditions, the sum $\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$ is called the *total differential of the function u* and is denoted by the symbol du.

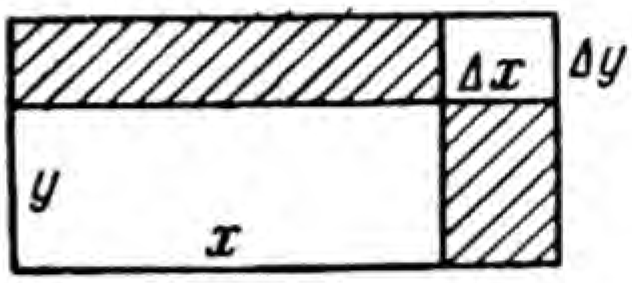

Fig. 153.

It is obviously the sum of the partial differentials of the function u with the given properties with respect to all of its independent variables:

$$\boxed{du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}\cdot dy} \qquad \text{(I)}$$

The concept of total differential can be extended to a function of more than 2 independent variables.

5°. **Example.** To find the total differential of the function

$$u=\operatorname{arc\,tan}\frac{x}{y}.$$

Solution. Differentiating with respect to x, we get

$$\frac{\partial u}{\partial x}=\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{1}{y}=\frac{y}{x^2+y^2}.$$

Differentiating with respect to y, we get

$$\frac{\partial u}{\partial y}=-\frac{x}{x^2+y^2}.$$

The total differential is

$$du=\frac{y\,dx}{x^2+y^2}-\frac{x\,dy}{x^2+y^2}=\frac{y\,dx-x\,dy}{x^2+y^2}.$$

6°. **Example.** Find the exact and approximate increase in the volume of a cylindrical tank when its radius increases from 2 to 2.01 metres and its height increases from 4 to 4.02 metres.

If the tank is made of 1-cm-thick sheet steel, the problem can also be stated thus: knowing the internal dimensions of the tank, $r=2$ m and $h=4$ m, find the amount of sheet steel required for its construction.

Solution. Let us denote the volume of the cylinder by u, the radius by x and the height by y. Then $u=\pi x^2 y$ is a function of two variables. And the exact change in volume is

$$\Delta u=\pi\cdot 2.01^2\cdot 4.02-\pi\cdot 2^2\cdot 4=\pi(16.241202-16)=0.241202\pi.$$

The approximate change in volume is $du=2\pi xy\,dx+\pi x^2\,dy=$ $=\pi(2\cdot 2\cdot 4\cdot 0.01+2^2\cdot 0.02)=0.24\pi$.

The difference $\Delta u-du=0.001202$ is extremely insignificant. This is the reason why the total differential of a function is often used in approximate calculations of the values of functions of several variables.

7°. Let $x, y, \ldots, z$ be the approximate values of arguments obtained by measurement and let $x+\Delta x$, $y+\Delta y$, $\ldots$, $z+\Delta z$ be their real values. Then $x, y, \ldots, z$ determine the approximate value of the function u, while $x+\Delta x$, $y+\Delta y$, $\ldots$, $z+\Delta z$ determine the real value of $u+\Delta u$.

The absolute error in the function is $|\Delta u|$ and the relative error—for the given definition of the value of the function—is $\left|\frac{\Delta u}{u}\right|$.

Since the calculation of Δu is in most cases a difficult mathematical operation, while du can be found without any particular difficulty from the formulas of differentiation, the relative error is taken to be equal to $\left|\frac{du}{u}\right|$.

8°. **Example.** Find the error in calculating the volume of a rectangular parallelepiped if slight errors are made in measuring its edges.

Solution. Let the measurements give edges of length x, y, z and let $\Delta x, \Delta y, \Delta z$ denote the small errors in measurement.

A resultant error Δu will then appear in the calculation of the volume.

If the errors $\Delta x, \Delta y, \Delta z$ are sufficiently small, the error of calculation Δu may be roughly considered equal to du.

Since $u=x\cdot y\cdot z$,

$$du=y\cdot z\cdot dx+x\cdot z\,dy+x\cdot y\cdot dz.$$

Dividing termwise by $u=x\cdot y\cdot z$, we obtain

$$\left|\frac{du}{u}\right|\leqslant\left|\frac{dx}{x}\right|+\left|\frac{dy}{y}\right|+\left|\frac{dz}{z}\right|.$$

If the error in each measurement did not exceed, say, 1%, then the error in the calculated volume of the parallelepiped does not exceed 3%.

Sec. 138. Differentiation of an Implicit Function

1°. If the dependence of the function y on the argument x is expressed by the equation

$$F(x, y) = 0,$$

not solved for y, then y is called an *implicit function* of x.

We shall now show that the derivative of an implicit function y can be found without solving its equation, by assuming that in the particular range of values of x and y a derivative of the function exists; this finds expression in the fact that a tangent can be drawn to the curve $F(x, y) = 0$ at the point x. Let us regard $F(x, y)$ as a function of $u = F(x, y)$, all values of which are equal to zero. Then the total differential is

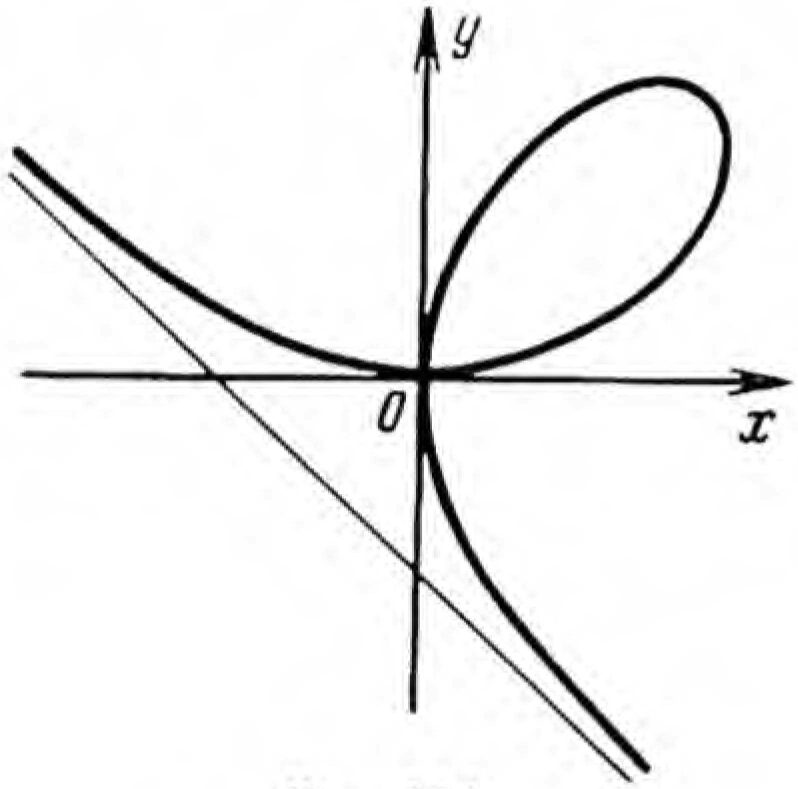

Fig. 154.

$$du = \frac{\partial F}{\partial x} \cdot dx + \frac{\partial F}{\partial y} \cdot dy = 0.$$

It is equal to zero because all values of $u = 0$; consequently, all values of $\Delta u = 0$. Dividing this equation termwise by dx, we have

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0.$$

Whence we find the derivative $\frac{dy}{dx}$:

$$\boxed{\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}} \qquad \text{(II)}$$

2°. **Example.** Find the derivative of the function y defined by the equation

$$x^3 + y^3 - 3axy = 0.$$

This equation defines a curve called the folium of Descartes (Fig. 154). By formula (II) we have

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{3x^2 - 3ay}{3y^2 - 3ax} = \frac{ay - x^2}{y^2 - ax}.$$

3°. **Example.** Find the equation of a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1).

Solution. The equation of a tangent is

$$y-y_1=\frac{dy_1}{dx_1}(x-x_1).$$

Let us find $\frac{dy}{dx}$ from the equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ by formula (II) without solving the equation for y:

$$\frac{dy}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}=-\frac{\frac{2x}{a^2}}{\frac{2y}{b^2}}=-\frac{b^2x}{a^2y};\quad \frac{dy_1}{dx_1}=-\frac{b^2x_1}{a^2y_1}.$$

Substituting the value of $\frac{dy_1}{dx_1}$ into the equation of the tangent, we find

$$y-y_1=-\frac{b^2x_1}{a^2y_1}(x-x_1),$$

or

$$a^2(yy_1-y_1^2)+b^2(xx_1-x_1^2)=0.$$

Dividing termwise by a^2b^2, we get

$$\frac{yy_1}{b^2}-\frac{y_1^2}{b^2}+\frac{xx_1}{a^2}-\frac{x_1^2}{a^2}=0.$$

Whence

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1.$$

Consequently,

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1,$$

which is the equation of the tangent to the ellipse.

4°. Formula (II) derived for obtaining the derivative of an implicit function of one variable may be used to find the partial derivatives of an implicit function of several variables. For example, let the dependence of the function z on two independent variables x and y be expressed by the equation

$$F(x, y, z)=0,$$

not solved for z. Here the function z is an implicit function of two independent variables x and y. We shall assume that it has partial derivatives for the given range of values of x and y.

Treating y as a constant and x as the independent variable, we get, from formula (II),

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}.$$

Now treating x as a constant and y as the independent variable, we get, from formula (II),

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}.$$

CHAPTER XIII

EXPANSION OF A FUNCTION IN A POWER SERIES

Sec. 139. Definitions

1°. *The expression*

$$u_1+u_2+u_3+\ldots+u_n+\ldots,$$

is called a series, in which $u_1, u_2, u_3, \ldots, u_n, \ldots,$ *respectively called the first, second, third, etc., terms of the series, are numbers such that their values depend on their place in the series in accordance with some definite rule.*

The value of any term u_n is a function of n. An infinite series is defined by a formula or rule that gives the value of any term of the series depending upon its position in the series. Thus an infinitely decreasing geometric progression is defined if its first term a and the common ratio q are known, since any nth term of the series can be found from the formula: $u_n=a\cdot q^{n-1}$.

2°. If we add the first n terms of an infinite series u_1+u_2+ $+u_3+\ldots+u_n+\ldots,$ the sum s_n,

$$s_n=u_1+u_2+u_3+\ldots+u_n,$$

of the first n terms of the series is called the nth order partial (or particular) sum. The value of a partial sum varies with the value of n. *If the number of terms n increases indefinitely and the partial sum s_n has the limit s*

$$\lim_{n\to\infty} s_n=s,$$

then s is called the sum of the infinite series, and the series itself is termed convergent. If s_n has no limit, the series is termed divergent.

The series $a+aq+aq^2+\ldots+aq^n+\ldots,$ where $|q|<1$ is convergent since its sum $s=\frac{a}{1-q}$. We shall call it a geometric series.

The series $1+1+1+\ldots+1+\ldots$ is divergent since its sum s_n, equal to the number of terms n, increases without bound.

The series $1-1+1-1+1-1+\dots$ is also divergent since here the sum s_n is either equal to 1 or equal to 0, and does not tend to a limit as n increases indefinitely.

Sec. 140. Necessary Condition for Convergence

1°. For any series

$$s_n = u_1 + u_2 + u_3 + \dots + u_{n-1} + u_n,$$

$$s_{n-1} = u_1 + u_2 + u_3 + \dots + u_{n-1}.$$

Subtracting, we have

$$s_n - s_{n-1} = u_n.$$

If the series is convergent, s_n and s_{n-1} tend to the same number s, its limit, as n increases without bound, and therefore their difference u_n tends to zero.

Thus, *in a convergent series* $\lim\limits_{n\to\infty} u_n = 0$.

Therefore, if the nth term of a series does not tend to zero as n increases indefinitely, the series diverges.

2°. This is not a sufficient condition.

Example. The series

$$1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}+\dots$$

is called harmonic. It is clear that the nth term of the harmonic series $\frac{1}{n}$ tends to zero if n increases without bound.

We shall show that this series, however, diverges. First rewrite the series as

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+ \\ +\left(\frac{1}{9}+\frac{1}{10}+\dots+\frac{1}{16}\right)+\dots \qquad (1)$$

then take the following series:

$$\frac{1}{2}+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+ \\ +\left(\frac{1}{16}+\frac{1}{16}+\dots+\frac{1}{16}\right)+\dots \qquad (2)$$

In the second series, each sum in the brackets is equal to $\frac{1}{2}$. And so the series may be rewritten as

$$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\dots+\frac{1}{2}+\dots \qquad (3)$$

If the number of terms of series (3) increases indefinitely, their sum also increases without bound. The terms of the harmonic

series are greater than the corresponding terms of series (3) or equal to them; hence the sum of the terms of the harmonic series will the more so increase indefinitely. And therefore the harmonic series is divergent.

Sec. 141. Conditional and Absolute Convergence

1°. Let us take the series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots$$

Let us examine the sum of its first $2n$ members, representing the sum as

$$s_{2n}=\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\ldots+\left(\frac{1}{2n-1}-\frac{1}{2n}\right) \quad (1)$$

and also as

$$s_{2n}=1-\left[\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{6}-\frac{1}{7}\right)+\ldots+\right.$$
$$\left.+\left(\frac{1}{2n-2}-\frac{1}{2n-1}\right)+\frac{1}{2n}\right]. \quad (2)$$

Every bracketed difference is positive.

Hence from (1) it follows that the sum s_{2n} is positive and increases with n; and from (2) it follows that the sum does not exceed the first term, 1.

Consequently, when $n\to\infty$ the sum of an even number of terms s_{2n}, as an increasing bounded quantity, has the limit $\lim\limits_{n\to\infty} s_{2n}=S$ (Sec. 58, 2°).

Let us consider the sum of an odd number of terms s_{2n-1}.

Since $s_{2n-1}-\frac{1}{2n}=s_{2n}$, $s_{2n-1}=s_{2n}+\frac{1}{2n}$ and $\lim\limits_{n\to\infty} s_{2n-1}=\lim\limits_{n\to\infty} s_{2n}+$ $+\lim\limits_{n\to\infty}\frac{1}{2n}=s+0=s$; in other words, both sums, s_{2n-1} and s_{2n}, have the same limit s when $n\to\infty$.

Whence it follows that s_n, which is alternately the sum of an odd and even number when $n\to\infty$, has a limit s, and the series is convergent.

Let us take the series made up of the absolute values of the terms of the previous series:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{2n}+\ldots$$

This series is harmonic and, as we know, divergent. This means that the given series converges, while the series made up of the absolute values of its terms diverges.

2°. **Definitions.** *A series converges conditionally, or is said to be conditionally convergent, if it converges, while the absolute values of its terms constitute a divergent series.* The foregoing series is conditionally convergent.

A series converges absolutely, or is said to be absolutely convergent, if the absolute values of its terms constitute a series which converges.

We shall show in Sec. 142 that the given series is convergent if the absolute values of its terms constitute a convergent series.

3°. The series previously studied $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\dots$ has alternating signs. Such a series (alternating series) has the form

$$u_1-u_2+u_3-u_4+\dots+(-1)^{n-1}\cdot u_n+\dots,$$

where $u_1, u_2, u_3, \dots, u_n, \dots$ are positive numbers.

Leibniz condition. *If the absolute value of a term of an alternating series* $u_1-u_2+u_3-u_4+\dots+(-1)^{n-1}\cdot u_n+\dots$ *decreases with increasing n so that* $\lim\limits_{n\to\infty} u_n=0$, then

1) *the series is convergent,*

2) *the sum of the series does not exceed the magnitude of the first term, i.e.,* $s\leqslant u_1$. Proof is similar to that employed in 1°.

4°. If all terms beginning with the $(n+1)$st term of an alternating infinite series are deleted and the partial sum s_n is taken for the sum of the infinite series, the absolute error thus commited does not exceed the absolute value of u_{n+1}.

Indeed, all the deleted terms form an alternating infinite series, of which u_{n+1} is the first term. Hence the sum of this series, in absolute value, does not exceed $|u_{n+1}|$.

Sec. 142. Comparison Theorem and d'Alembert's Test

1°. **Comparison theorem.** *If the terms of a series* $u_1+u_2+u_3+\dots+u_n+\dots$ *do not exceed, in absolute value, the corresponding terms of a convergent series with positive terms* $v_1+v_2+\dots+v_n+\dots$, *then the given series converges absolutely.*

This is clear from the following. It is given that all the terms of the series $v_1+v_2+v_3+\dots+v_n+\dots$ are positive. Let us take a series made up of the absolute values of the first series:

$$|u_1|+|u_2|+|u_3|+\dots+|u_n|+\dots.$$

All its terms are also positive. Since $|u_n|\leqslant v_n$ for any n, the sum s_n of the absolute values of the first n terms of the (u) series does not exceed the sum σ_n of the first terms of the (v) series,

$$s_n\leqslant\sigma_n.$$

It is given that the (v) series converges, therefore

$$\lim_{n=\infty}\sigma_n=\sigma.$$

When n increases without bound, the sum s_n of the absolute values of the (u) series increases all the time but always remains less than the positive number σ. Hence (Sec. 56, 2°) it has a limit.

$\lim_{n\to\infty} s_n = s$, where $s \leqslant \sigma$, which means that the (u) series converges absolutely.

Example. Prove the convergence of the series $1+\frac{1}{2^2}+\frac{1}{3^3}+ + \ldots + \frac{1}{n^n}+\ldots$

Proof. No term of the given series exceeds the corresponding term of the infinitely decreasing geometric progression

$$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}+\ldots$$

and so the given series converges like a geometric series.

2°. Let us prove that *a given series converges if the absolute values of its terms constitute a convergent series.*

Let the series

$$u_1+u_2+u_3+\ldots+u_n+\ldots, \tag{1}$$

the terms of which are positive and negative, converge absolutely. Let us make two new series: one, out of the positive terms of the given series

$$u_1'+u_2'+u_3'+\ldots+u_n'+\ldots, \tag{2}$$

and the other, out of the absolute values of the negative terms of the given series:

$$u_1''+u_2''+u_3''+\ldots+u_n''+\ldots \tag{3}$$

Each of these two series converges since their terms do not exceed th corresponding terms of the convergent series

$$|u_1|+|u_2|+|u_3|+\ldots+|u_n|+\ldots,$$

composed of the absolute values of the given series (1).

The sum of the given series (1) is equal to the difference between the sums of series (2) and (3).

3°. D'Alembert's test is as follows:*

If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1$, *the series converges and converges absolutely; if* $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|>1$, *the series diverges.*

If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=1$, *the question as to the convergence of the series remains undecided and the series may be either convergent or divergent.*

Example. The series $1+\frac{1}{1}+\frac{1}{1\cdot2}+\frac{1}{1\cdot2\cdot3}+\ldots$ is convergent because $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{1}{n}\right|=0$.

* We give this test without proof.

4°. Note that the comparison theorem (test) and d'Alembert's test constitute *sufficient conditions for the absolute convergence of a series*, i.e., if the conditions are satisfied, it may be concluded that the series is absolutely convergent. But these are not necessary conditions for the absolute convergence of a series, for there may be absolutely convergent series which do not fulfil these conditions.

Sec. 143. Power Series and the Condition for Its Convergence

1°. **Definition:** *A power series is called a series of the type*

$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots,$$

where a_0, a_1, a_2, ... a_n represent constant numbers called coefficients of the series.

A power series is completely known if the sequence of its coefficients is known.

2°. For a power series, we may have one of the three following cases.

1. A power series may diverge for all values of x (except $x=0$) and converge for $x=0$. For example, the series

$$1 + 1\cdot x + 1\cdot 2\cdot x^2 + 1\cdot 2\cdot 3\cdot x^3 + \ldots + 1\cdot 2\cdot 3\ldots nx^n + \ldots$$

diverges for all values of x except $x=0$, since by d'Alembert's test

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}|nx| = \infty.$$

For $x=0$ all the terms of the series, except the first term, a_0, are zero, therefore the series converges.

2. A power series may converge absolutely for all values of x. For example, the series

$$1 + \frac{x}{1} + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \ldots + \frac{x^n}{1\cdot 2\cdot 3\ldots n} + \ldots$$

converges absolutely for all values of x without exception, since by d'Alembert's test

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x}{n}\right| = \lim_{n\to\infty}\left|x\cdot\frac{1}{n}\right| = 0.$$

3. A power series may converge for some and diverge for other values of x. For example the series

$$1 + x + x^2 + \ldots + x^n + \ldots$$

is convergent for $|x| < 1$ and divergent for $|x| \geqslant 1$, since in the former case the series is an infinitely decreasing geometric pro-

gression, while in the latter case the sum of its terms increases indefinitely.

3°. **Convergence test of a power series.** *If a power series converges for some value of $x = x_0 \neq 0$, then it converges absolutely for any value of x whose absolute value is smaller than the absolute value of x_0*, i.e.,

$$|x| < |x_0|\,^*.$$

4°. Note that if there exists a value $x = x_0$ for which the series diverges, then the series will diverge for any value of x for which $|x| > |x_0|$. Indeed, if the series converged for such a value of x, then, on the basis of the convergence test, the series should also converge for x_0, which is less than x in absolute value.

Sec. 144. Differentiation of a Power Series

Let the power series $a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$ converge for $x = x_0$. From the foregoing the series converges for all other values of x for which $|x| < |x_0|$. Hence for each value of x in the interval $0 \leqslant |x| < |x_0|$, just as for $x = x_0$, the sum of the series has a definite value. For this reason the sum of the series is a function of x. Let us denote it by $f(x)$. Then

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots$$

In the theory of series it is proved that *any function $f(x)$ expressed by the power series*

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots,$$

possesses derivatives of all orders on the interval of convergence, and these derivatives may be obtained by term-by-term differentiation of the given series:

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots;$$
$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1} + \ldots;$$
$$f''(x) = 1\cdot 2a_2 + 2\cdot 3a_3x + \ldots + (n-1)\,n\cdot a_nx^{n-2} + \ldots;$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$f^{(k)}(x) = 1\cdot 2\ldots ka_k + 2\cdot 3\ldots(k+1)\,a_{k+1}x + \ldots +$$
$$+ (n-k+1)(n-k+2)\ldots na_nx^{n-k} + \ldots, \text{ etc.}$$

Sec. 145. Maclaurin's Series

1°. Assume the function $f(x)$ is expressed by the power series $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$, i.e.,

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots$$

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* This test is given without proof.

This series will be known if its coefficients are known. To find the coefficients, differentiate the given series:

$$f'(x) = 1\cdot a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1} + \ldots;$$
$$f''(x) = 1\cdot 2a_2 + 2\cdot 3a_3x + \ldots + (n-1)\cdot n\cdot a_nx^{n-2} + \ldots;$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$f^{(k)}(x) = 1\cdot 2\ldots ka_k + 2\cdot 3\ldots(k+1)\,a_{k+1}x + \ldots + (n-k+1)(n-k+2)\ldots na_nx^{n-k} + \ldots$$

Putting $x=0$ in these series, we get

$$f(0) = a_0,$$
$$f'(0) = a_1,$$
$$f''(0) = 1\cdot 2\cdot a_2,$$
$$\ldots\ldots\ldots$$
$$f^{(k)}(0) = 1\cdot 2\cdot 3\ldots k\cdot a_k, \text{ etc.}$$

From these equalities we find the coefficients of the power series:

$$a_0 = f(0),$$
$$a_1 = \frac{f'(0)}{1},$$
$$a_2 = \frac{f''(0)}{1\cdot 2},$$
$$\ldots\ldots\ldots$$
$$a_k = \frac{f^{(k)}(0)}{1\cdot 2\cdot 3\ldots k},$$
$$\ldots\ldots\ldots$$

Consequently, if a function $f(x)$ is expressed by a power series, there is only one such series, namely:

$$\boxed{f(x) = f(0) + \frac{x}{1}\cdot f'(0) + \frac{x^2}{1\cdot 2}f''(0) + \frac{x^3}{1\cdot 2\cdot 3}f'''(0) + \ldots + \frac{x^n}{1\cdot 2\cdot 3\ldots n}f^{(n)}(0) + \ldots}$$

This series is customarily known as the Maclaurin series.

2°. **Example.** Expand the function e^x into a Maclaurin series.

Solution. Putting $f(x) = e^x$ and differentiating, we find that

$$f'(x) = e^x,\ f''(x) = e^x,\ \ldots,\ f^{(n)}(x) = e^x,\ \ldots$$

When $x=0$, we have

$f(0) = e^0 = 1$ and also $f'(0) = f''(0) = \ldots = f^{(n)}(0) = \ldots = e^0 = 1.$

The Maclaurin series turns into the following series:

$$1 + x + \frac{x^2}{1\cdot 2} + \frac{x^2}{1\cdot 2\cdot 3} + \ldots + \frac{x^n}{1\cdot 2\cdot 3\ldots n} + \ldots$$

As was shown in Sec. 143, this series converges for all values of x without exception.

3°. Not every function can be expanded into a Maclaurin series. Take, for example, the function $f(x) = \ln x$. We may differentiate it as many times as we please: $f'(x) = \frac{1}{x}$; $f''(x) = -\frac{1}{x^2}$; $f'''(x) = \frac{2}{x^3}$, etc. And for all values of x other than zero the function itself and all its derivatives have definite numerical values. But at $x = 0$, there are no definite numerical values of the coefficients of the series. For this reason it is impossible to expand $\ln x$ in Maclaurin's series.

Sec. 146. Taylor's Series

Very often the function $f(x)$ is expanded into a power series not in powers of x but in powers of the difference $x - a$, where a is some constant for which both the series and all its derivatives are definite numbers.

Let us denote $x - a$ by h:

$$x - a = h.$$

Then

$$x = a + h,$$

$$f(x) = f(a + h).$$

Since a is a constant number, h appears as a new variable such that $f(x)$ is a function of h. Let us denote it by $\varphi(h)$:

$$\varphi(h) = f(a + h).$$

Differentiating this equality, we get

$$\varphi'(h) = f'(a + h),\ \varphi''(h) = f''(a + h),\ \ldots$$
$$\varphi^{(n)}(h) = f^{(n)}(a + h).$$

At $h = 0$, we obtain

$$\varphi(0) = f(a),$$
$$\varphi'(0) = f'(a),$$
$$\varphi''(0) = f''(a),$$
$$\cdots\cdots\cdots\cdots$$
$$\varphi^{(n)}(0) = f^{(n)}(a), \text{ etc.}$$

Let us write the Maclaurin series for the function $\varphi(h)$:

$$\varphi(h) = \varphi(0) + \frac{h}{1} \cdot \varphi'(0) + \frac{h^2}{1 \cdot 2} \varphi''(0) + \frac{h^3}{1 \cdot 2 \cdot 3} \varphi'''(0) + \ldots +$$
$$+ \frac{h^n}{1 \cdot 2 \cdot 3 \ldots n} \varphi^{(n)}(0) + \ldots$$

But

$$\varphi(h)=f(x),\quad h=x-a,\quad \varphi(0)=f(a),\quad \varphi'(0)=f'(a),\ \ldots$$

Making substitutions into the Maclaurin series in accordance with these equalities, we obtain

$$\boxed{\begin{aligned} f(x)=f(a)+\frac{x-a}{1}f'(a)+\frac{(x-a)^2}{1\cdot 2}f''(a)+\ldots+ \\ +\frac{(x-a)^n}{1\cdot 2\ldots n}f^{(n)}(a)+\ldots \end{aligned}}$$

This is the Taylor series. The Maclaurin series is a special case of this series for $a=0$.

Sec. 147. Convergence of the Taylor Series and the Maclaurin Series

1°. Let us ask: what error will be committed if in an expansion of the function $f(x)$ in a Taylor-Maclaurin series we put the value of the function $f(x)$ equal to the sum of the first n terms of the series, thus ignoring the sum of all terms $n+1$ and above. The sum of all these terms, commencing with $n+1$, is designated by R_n. By disregarding the sum of all terms of the series from $n+1$, we commit an error equal to R_n. Lagrange, Cauchy and others found an expression for R_n.

According to Lagrange,

$$R_n=\frac{x^n}{1\cdot 2\ldots n}f^{(n)}(\xi)\ \text{(for the Maclaurin series);}$$

$$R_n=\frac{(x-a)^n}{1\cdot 2\cdot 3\ldots n}f^{(n)}(\xi)\ \text{(for the Taylor series),}$$

where ξ is some number intermediate between 0 and the value of x for the Maclaurin series and between a and x for the Taylor series.

The formula does not give a definite value of ξ; this makes it difficult to use.

2°. When expanding the function $f(x)$ into a series, one has to determine whether the given series converges to the function $f(x)$, in other words, *whether $f(x)$ is the limit of the partial sum s_n of the first n terms of the series as n increases without limit.*

Putting $f(x)=s_n+R_n$, we find that

$$f(x)=\lim_{n\to\infty} s_n,\quad \text{if}\quad \lim_{n\to\infty} R_n=0,\quad \text{i.e.,}$$

if, for the given values of x, R_n tends to zero when n increases indefinitely, then $f(x)$ is the sum of the series for these values of x.

If, for the given values of x, R_n tends to a limit different from zero, then the series converges but not to the function $f(x)$; and if R_n does not tend to any limit or increases without bound, the series diverges.

3°. A practical rule for expanding functions in series is the following: *if in the interval* $|x| < |x_0|$ *the derivatives of all orders of the function* $f(x)$ *are less in absolute value than some positive number* M, *then, within this interval, the Maclaurin series converges to the given function.*

Proof. It is given that

$$|f^{(n)}(x)| < M.$$

Substituting M for the derivative appearing in the Lagrange formula for R_n, we get

$$|R_n| < M \cdot \frac{|x^n|}{1 \cdot 2 \cdot 3 \ldots n}.$$

Let us take a series whose general term is the right-hand side of this inequality:

$$M + M\frac{|x|}{1} + M\frac{|x^2|}{1 \cdot 2} + M\frac{|x^3|}{1 \cdot 2 \cdot 3} + \ldots + M\frac{|x^n|}{1 \cdot 2 \cdot 3 \ldots n} + \ldots$$

Taking the common factor outside the brackets, we conclude (Sec. 143) that the series converges; therefore its nth term,

$$M\frac{|x^n|}{1 \cdot 2 \cdot 3 \ldots n},$$

tends to zero. It follows that R_n also tends to zero and the series converges to the given function.

Sec. 148. Examples of Expanding Functions in Powers of x. Binomial Series

1°. **Expansion of** $\sin x$.

Putting $f(x) = \sin x$, we find that

$$f'(x) = \cos x,$$

$$f''(x) = -\sin x,$$

$$f'''(x) = -\cos x,$$

$$f^{\mathrm{IV}}(x) = \sin x, \text{ etc.}$$

When $x = 0$ we have

$$f(0) = \sin 0 = 0; \quad f'(0) = \cos 0 = 1; \quad f''(0) = -\sin 0 = 0;$$

$$f'''(0) = -\cos 0 = -1; \quad f^{\mathrm{IV}}(0) = \sin 0, \text{ etc.}$$

The Maclaurin series will be written as follows:

$$\boxed{\sin x = \frac{x}{1} - \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{x^7}{1\cdot 2\cdot 3\ldots 7} + \ldots + (-1)^n \cdot \frac{x^{2n-1}}{1\cdot 2\cdot 3\ldots (2n-1)} + \ldots} \qquad (1)$$

Since the derivatives do not exceed 1 for any value of x, series (1) converges to the function $f(x) = \sin x$ for all values of x.

2°. **Expansion** of $\ln(1+x)$.

We have already seen (Sec. 145) that expansion of $\ln x$ in a Maclaurin series is impossible since the function $\ln x$ and all its derivatives lose numerical significance at $x=0$. But the function $\ln(1+x)$ may be expanded into a Maclaurin series.

Putting $f(x) = \ln(1+x)$, we find that

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = -\frac{1}{(1+x)^2}, \quad f'''(x) = \frac{1\cdot 2}{(1+x)^3},$$

$$f^{\mathrm{IV}}(x) = -\frac{1\cdot 2\cdot 3}{(1+x)^4}, \ \ldots \ f^{(n)}(x) = (-1)^{n-1}\cdot\frac{1\cdot 2\cdot 3\ldots(n-1)}{(1+x)^n}, \ \ldots$$

At $x=0$ we have

$$f(0) = \ln 1 = 0, \quad f'(0) = 1, \quad f''(0) = -1,$$

$$f'''(0) = 1\cdot 2, \quad f^{\mathrm{IV}}(0) = -1\cdot 2\cdot 3, \ \ldots,$$

$$f^{(n)}(0) = (-1)^{n-1}\cdot 1\cdot 2\cdot 3\ldots(n-1), \ \ldots$$

The Maclaurin series will be written as follows:

$$\boxed{\ln(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n-1}\cdot\frac{x^n}{n} + \ldots} \qquad (2)$$

To determine the interval of convergence of this series we apply the d'Alembert test.

According to the d'Alembert test, series (2) converges if

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1} : \frac{x^n}{n}\right| = \lim_{n\to\infty}\left|x\cdot\frac{n}{n+1}\right| =$$

$$= |x|\cdot\lim_{n\to\infty}\left|\frac{n}{n+1}\right| < 1.$$

Even for an indefinite increase in n the fraction $\frac{n}{n+1} = \frac{n+1-1}{n+1} = 1 - \frac{1}{n+1}$ will tend towards the limit 1. Hence, for the above-written inequality to hold, the absolute value of x must not exceed 1.

Note that if $x=1$ the expansion of $\ln(1+x)$ takes the form of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$, which is convergent

(Sec. 141). According to the convergence test of a power series, the given series (2) converges for all values of x for which $|x|<1$. At $x=-1$ the expansion is reduced to the series $-1-\frac{1}{2}-$ $-\frac{1}{3}-\ldots$ which is divergent since it is a harmonic series multiplied by -1.

Thus, series (2) converges in the interval $-1<x\leqslant+1$. It will be noted that in this interval, $\lim\limits_{n\to\infty} R_n=0$.

3°. **Binomial series.** We have $f(x)=(1+x)^m$. Differentiating, we get

$$f'(x)=m(1+x)^{m-1},$$
$$f''(x)=m(m-1)(1+x)^{m-2},$$
$$f'''(x)=m(m-1)(m-2)(1+x)^{m-3}, \text{ etc.}$$

At $x=0$, we have

$$f(0)=1,\quad f'(0)=m,\quad f''(0)=$$
$$=m(m-1),\quad f'''(0)=m(m-1)(m-2),\quad \text{etc.}$$

The Maclaurin series will be written as follows:

$$\boxed{(1+x)^m=1+mx+\frac{m(m-1)}{1\cdot2}x^2+ +\frac{m(m-1)(m-2)}{1\cdot2\cdot3}x^3+\ldots} \tag{3}$$

This is called a *binomial series*. It is an infinite series only when m is not a natural number and is not equal to zero, since it is only in this case that none of the factors m, $(m-1)$, $(m-2)$, $(m-3)$, ... becomes zero.

According to the d'Alembert test, series (3) converges if

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{m(m-1)\ldots(m-n+2)(m-n+1)}{1\cdot2\cdot3\ldots(n-1)n}\times\right.$$
$$\left.\times x^n:\frac{m(m-1)\ldots(m-n+2)}{1\cdot2\cdot\ldots(n-1)}x^{n-1}\right|=$$
$$=\lim_{n\to\infty}\left|\frac{m-n+1}{n}x\right|=|x|\cdot\lim_{n\to\infty}\left|\frac{m+1}{n}-1\right|<1.$$

Since with $m=$ constant and $n\to\infty$ the limit $\frac{m+1}{n}$ is zero,

$$\lim_{n\to\infty}\left|\frac{m+1}{n}-1\right|=1.$$

Hence, for the foregoing inequality to hold it is necessary that $|x|<1$. Consequently, a binomial series converges for values of x for which $|x|<1$, and the series can be used for all values of x

whose absolute values do not exceed 1. It will be noted that under these conditions $\lim_{n\to\infty} R_n = 0$.

Let us consider some *particular cases* of the binomial series.

1. Let $m = -1$. Then

$$(1+x)^{-1} = 1 + (-1)\,x + \frac{(-1)\cdot(-2)}{1\cdot 2}\,x^2 + \\ + \frac{(-1)(-2)(-3)}{1\cdot 2\cdot 3}\,x^3 + \ldots,$$

or

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots$$

2. Let $m = \frac{1}{2}$. Then

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}\,x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1\cdot 2}\,x^2 + \\ + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1\cdot 2\cdot 3}\,x^3 + \ldots,$$

or

$$\sqrt{1+x} = 1 + \frac{1}{2}\,x - \frac{1}{8}\,x^2 + \frac{1}{16}\,x^3 + \ldots$$

3. Applying the above expansion for $m = -\frac{1}{2}$, we find that

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}\,x + \frac{3}{8}\,x^2 - \frac{5}{16}\,x^3 + \ldots$$

4. If the exponent m is a natural number, for example, $m = 4$, then $m - 4 = 0$, and so all the terms of the series containing this factor will be equal to zero and the series will be finite.

Indeed, for $m = 4$,

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{1\cdot 2}\,x^2 + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}\,x^3 + \\ + \frac{m(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3\cdot 4}\,x^4.$$

Every succeeding term becomes zero because it contains the factor $m - 4 = 0$.

This series is Newton's well-known binomial formula of algebra.

Sec. 149. Calculations by Means of Series

1°. Evaluate *e*.

In the expanded function of e^x (Sec. 145)

$$e^x = 1 + x + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \ldots + \frac{x^n}{1\cdot 2\cdot 3\ldots n} + \ldots$$

let us put $x=1$. Then

$$e=1+1+\frac{1}{1\cdot2}+\frac{1}{1\cdot2\cdot3}+\ldots+\frac{1}{1\cdot2\cdot3\ldots n}+\ldots$$

This is the formula for e which we introduced in Sec. 91. This series enables us to find e to a very high degree of accuracy using a comparatively small number of terms of the series.

For example, confining ourselves to the first ten terms of the series, the error is equal to the sum of the series

$$\frac{1}{1\cdot2\cdot3\ldots9\cdot10}+\frac{1}{1\cdot2\cdot3\ldots10\cdot11}+\frac{1}{1\cdot2\cdot3\ldots10\cdot11\cdot12}+\ldots$$

Taking the first term of this series out of the brackets, we get

$$\frac{1}{1\cdot2\cdot3\ldots9\cdot10}\cdot\left(1+\frac{1}{11}+\frac{1}{11\cdot12}+\frac{1}{11\cdot12\cdot13}+\ldots\right).$$

For terms within the brackets, each factor in the denominator of each fraction is greater than 10. If we substitute 10 for each such factor, the denominator in each case will decrease and the fraction will increase, and the error will be less than

$$\frac{1}{1\cdot2\cdot3\ldots9\cdot10}\left(1+\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\ldots\right).$$

Within the brackets is the sum of an infinitely decreasing geometric progression equal to

$$\frac{a}{1-q}=\frac{1}{1-\frac{1}{10}}=\frac{10}{9}.$$

Consequently, the error is less than

$$\frac{1}{1\cdot2\cdot3\ldots9\cdot10}\cdot\frac{10}{9}=\frac{1}{9\cdot1\cdot2\cdot3\ldots9}=\frac{1}{3265920}<0.000001.$$

2°. **Evaluation** of $\sin x$. If in the expansion

$$\sin x=x-\frac{x^3}{1\cdot2\cdot3}+\frac{x^5}{1\cdot2\cdot3\cdot4\cdot5}-\frac{x^7}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}+\ldots$$

we take only the sum of the first n terms, the absolute value of the error thus committed will not exceed the value of the next term $n+1$ (Sec. 141, 4°).

Let us calculate $\sin 1°$ and $\sin 10°$ to an accuracy of 0.00001. In the expansion of $\sin x$, x represents the value of the arc in radian measure. For an angle of $1°$, $x=\frac{\pi}{180°}=0.017453$. The second term of the expansion $\frac{x^3}{6}<\frac{(0.02)^3}{6}=\frac{0.000008}{6}<0.000002$. Therefore, all terms from and after the second may be deleted and the first five figures in the value of $\sin 1°$ are given by the first term of the series: $\sin 1°=0.01745$.

For the arc 10°, $x=\frac{\pi}{18}=0.174533$. Here, the second term of the expansion $\frac{x^3}{6}>\frac{(0.1)^3}{6}>0.0001$, and it cannot be rejected. But the third term $\frac{x^5}{120}<\frac{(0.2)^5}{120}=\frac{0.00032}{120}<0.000003$. Hence the third term and all the succeeding ones can be ignored. Thus, to an accuracy up to 0.00001,

$$\sin 10°=x-\frac{x^3}{1\cdot 2\cdot 3}=0.174533-0.000886=0.17365.$$

It is clear that for a larger arc more terms of the series would have to be taken so as to maintain the required accuracy.

Let us confine ourselves to two terms of the series

$$\sin x=x-\frac{x^5}{6}$$

and try to find out for which values of x this assumption holds to an accuracy of 0.00001. The error in such cases may be taken equal to $\frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}=\frac{x^5}{120}$. In fact, it is even less than the value of this term of the series.

$$\text{Then } \frac{x^5}{120}<0.00001;\ x^5<0.0012;\ x<\sqrt[5]{0.0012}=0.2605.$$

$$x=14.9°$$

In practical calculations no more than three terms of a series are used since it is more convenient to employ formulas of trigonometry than a larger number of terms.

3°. **Calculation** of logarithms.

We have (Sec. 148. 2°)

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\ldots; \tag{1}$$

giving x all possible numerical values between 0 and 1, we can calculate the logarithms of all the numbers between 1 and 2. At $x=1$

$$\ln(1+x)=\ln 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots$$

(the series considered in Sec. 141). Thus, ln 2 can be calculated to any degree of accuracy.

The natural logarithms of other integral numbers can be calculated, for example, as follows:

Assuming in formula (1)

$$x=\frac{1}{p},$$

we get on the left-hand side of formula (1)

$$\ln(1+x)=\ln\left(1+\frac{1}{p}\right)=\ln\frac{p+1}{p}=\ln(p+1)-\ln p,$$

and on the right-hand side

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots=\frac{1}{p}-\frac{1}{2p^2}+\frac{1}{3p^3}-\frac{1}{4p^4}+\ldots$$

Whence

$$\ln(p+1)=\ln p+\frac{1}{p}-\frac{1}{2p^2}+\frac{1}{3p^3}-\frac{1}{4p^4}+\ldots$$

If we put $p=2$, we find

$$\ln 3=\ln 2+\frac{1}{2}-\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}-\frac{1}{4\cdot 2^4}+\ldots$$

Logarithms can also be calculated by the use of other series, for example, by the expansion of $\ln\frac{1+x}{1-x}$. We shall not consider this point further beyond noting that a series is really useful only for calculating the logarithms of prime numbers. The logarithms of compound numbers can be found more simply and quickly by the use of rules for taking logarithms.

For example:

$$\ln 4=\ln(2^2)=2\cdot\ln 2;\quad \ln 6=\ln(2\cdot 3)=\ln 2+\ln 3.$$

Also note that we have spoken about the calculation of *natural* logarithms. To get decimal logarithms, it is sufficient to multiply the natural logarithm by the modulus of the base, i.e., by $\frac{1}{\ln 10}=$ $=0.4342944819\ldots$.

Sec. 150. Examples of Expansion in Powers of the Difference $x-a$

1. Expand x^3-3x^2+7x-4 in powers of $x-1$.

Solution. Putting $f(x)=x^3-3x^2+7x-4$ and differentiating, we get

$$f'(x)=3x^2-6x+7,$$
$$f''(x)=6x-6,$$
$$f'''(x)=6,$$
$$f^{\mathrm{IV}}(x)=0.$$

We now find the coefficients of Taylor's series (Sec. 146):

$$f(x)=f(a)+\frac{x-a}{1}f'(a)+\frac{(x-a)^2}{1\cdot 2}f''(a)+\frac{(x-a)^3}{1\cdot 2\cdot 3}f'''(a)+\ldots$$

For the given expansion $a=1$. Therefore,

$$f(a)=f(1)=1;\quad f'(a)=f'(1)=4;\quad f''(a)=f''(1)=0;$$
$$f'''(a)=f'''(1)=6;\quad f^{\mathrm{IV}}(a)=f^{\mathrm{IV}}(1)=0.$$

All succeeding coefficients are also equal to zero. Therefore,

$$f(x)=x^3-3x^2+7x-4=1+4(x-1)+(x-1)^3.$$

2. Expand $\sqrt{x}$ in powers of the difference $x-4$.

Solution. $f(x)=\sqrt{x}$. Differentiating $f(x)=x^{\frac{1}{2}}$ and putting $a=4$, we find $f(4)=\sqrt{4}=2$,

$$f'(x)=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}};\quad f'(4)=\frac{1}{2\sqrt{4}}=\frac{1}{4}.$$

$$f''(x)=-\frac{1}{4}x^{-\frac{3}{2}}=-\frac{1}{4x\sqrt{x}};\quad f''(4)=-\frac{1}{4\cdot 4\sqrt{4}}=-\frac{1}{32};$$

$$f'''(x)=\frac{3}{8}x^{-\frac{5}{2}}=\frac{3}{8x^2\sqrt{x}};\quad f'''(4)=\frac{3}{8\cdot 4^2\sqrt{4}}=\frac{3}{256},\ \text{etc.}$$

Hence,

$$\sqrt{x}=2+\frac{x-4}{4}-\frac{(x-4)^2}{64}+\frac{(x-4)^3}{512}-\ldots.$$

It is evident that, with such a series, the root can be calculated to any degree of accuracy.

3. Expand $\cos x$ in powers of the difference $x-\frac{\pi}{3}$.

Solution. Differentiating $f(x)=\cos x$ and putting $a=\frac{\pi}{3}$, we find

$$f(x)=\cos x;\qquad f\left(\frac{\pi}{3}\right)=\cos\frac{\pi}{3}=\frac{1}{2};$$
$$f'(x)=-\sin x;\qquad f'\left(\frac{\pi}{3}\right)=-\sin\frac{\pi}{3}=-\frac{\sqrt{3}}{2};$$
$$f''(x)=-\cos x;\qquad f''\left(\frac{\pi}{3}\right)=-\cos\frac{\pi}{3}=-\frac{1}{2};$$
$$f'''(x)=\sin x;\qquad f'''\left(\frac{\pi}{3}\right)=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2};$$
$$f^{\mathrm{IV}}(x)=\cos x;\qquad f^{\mathrm{IV}}\left(\frac{\pi}{3}\right)=\cos\frac{\pi}{3}=\frac{1}{2},\ \text{etc.}$$

Hence

$$\cos x=\frac{1}{2}-\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)-\frac{1}{4}\left(x-\frac{\pi}{3}\right)^2+$$
$$+\frac{\sqrt{3}}{12}\left(x-\frac{\pi}{3}\right)^3+\frac{1}{48}\left(x-\frac{\pi}{3}\right)^4-\ldots.$$

E. PROBLEMS AND EXERCISES

Sec. 1. Method of Coordinates

1. Construct the points: a) (3, 4), (—2, 5), (—4, —1) and (1, —7), b) $\left(1\frac{3}{5}, -0.8\right)$, (—0.6, —1.2), $\left(-1.75, \frac{2}{3}\right)$ and (0.7, 1.1).

2. Given the point M (5, —2), find the coordinates of a point symmetrical to M: a) with respect to the x-axis, b) with respect to the y-axis.

3. If the coordinates of a point are $x = -a$, $y = b$, what are the coordinates of a point symmetrical to the given point: a) with respect to the axis of abscissas, b) with respect to the axis of ordinates?

4. Points A (2, 5) and B (—3, 2) are the end points of the segment AB. Find the length of the projection of the segment: a) on the axis of abscissas, b) on the axis of ordinates.

5. Show that two triangles are isosceles and right-angled if their vertices are represented by the points: a) (—3, 4), (4, 3), (0, 0), b) (—4, —2), (—3, 5) and (0, 1).

6. Show that a triangle with vertices A (—4, 3), B (0, 2) and C (2, —5) is an obtuse triangle.

7. Find the abscissa of the point M if its ordinate is 4 and its distance from another point N (1, —2) is equal to 10 units of length.

8. Find the point on the axis of ordinates which is at 5 units distance from another point A (—3, 1).

9. Find the point on the axis of ordinates equidistant from the origin and from the point (3, —5).

10. Find the point on the axis of abscissas equidistant from the points A (—1, 0) and B (7, —4).

11. Find the centre of a regular hexagon if two of its adjacent vertices are A (2, 0) and B (5, $3\sqrt{3}$).

12. Find the point equidistant from three given points: A (0, —6), B (1, 1) and C (7, —7).

13. Find the centre of the circumscribed circle of a triangle whose vertices are A (4, —2), B (5, —3) and C (—4, —6).

14. The vertices of a triangle are A (1, —5), B (5, 2), C (0, —3). Find the mid-points of its sides.

15. A segment whose end points are $A\ (-2, 3)$ and $B\ (4, -1)$ is divided into three equal parts. Find the coordinates of the points of division.

16. A segment whose end points are $A\ (3, 2)$ and $B\ (15, 6)$ is divided into five equal parts. Find the coordinates of the points of division.

17. Find points symmetrical about the origin for the points: a) (2, 0), b) (0, —3), c) (2, 5), d) (—3, 1).

18. A line is drawn from the point $A\ (-3, 1)$ to the point B (4, —2). Up to what point must the line be produced in the same direction to double its length?

19. A line is drawn from the point (0, —1) to the point (—4, 3). To what point must it be produced in the same direction so as to treble its length?

20. In a triangle with vertices $A\ (3, 7)$, $B\ (-4, 0)$ and C (1, —4) find the length of the median of the side AC.

21. Find the centre of gravity of a triangle whose vertices are $A\ (1, 4)$, $B\ (-5, 0)$ and $C\ (-2, -1)$.

22. Given a triangle with vertices at $A\ (2, 1)$, $B\ (6, 4)$ and C (—4, 9). Find the point of intersection of the bisector of angle A with the opposite side BC.

23. Find the vertices of a triangle knowing the mid-points of its sides:

$$M\left(-\frac{1}{2},\ 3\,\frac{1}{2}\right),\ N\left(-1\,\frac{1}{2},\ -2\right)\ \text{and}\ P\left(2,\ 1\,\frac{1}{2}\right).$$

24. The points $A\ (1, 0)$, $B\ (2, 1)$ and $C\ (3, -2)$ are three consecutive vertices of a parallelogram. Find the fourth vertex D.

25. Knowing two adjacent vertices (2, 0), (—3, 3) of a parallelogram and the point of intersection $N\ (-1, 0)$ of its diagonals, find the other two vertices.

26. Three parallel forces P_1, P_2, P_3 are respectively applied at the points $M_1\ (x_1, y_1)$, $M_2\ (x_2, y_2)$, $M_3\ (x_3, y_3)$; determine the coordinates of the point of application of their resultant.

27. Employ the formulas obtained in the preceding problem for the case of three forces $P_1 = 12$ kg, $P_2 = -18$ kg, and $P_3 = 6$ kg applied at points $M_1\ (3, -5)$, $M_2\ (4, 1)$ and $M_3\ (6, 0)$, respectively, and interpret the result obtained.

28. What angle is formed with the x-axis by a straight line passing through the points: a) $M\ (0, 2)$ and $N\ (2, 4)$, b) $M\ (2, 0)$ and $N\ (-4, 6)$, c) $M\ (1, -1)$ and $N\ (3, -1)$?

29. Show that the quadrilateral $ABCD$ with vertices at $A\ (2, 6)$, $B\ (5, 1)$, $C\ (-1, -6)$ and $D\ (-4, -1)$ is a parallelogram.

30. Show that the diagonals AC and BD in the quadrilateral $ABCD$ with vertices at $A\ (2, -5)$, $B\ (7, -3)$, $C\ (6, 1)$ and D (—2, 3) are mutually perpendicular.

31. What is the relation between the slopes of two straight lines symmetrical about: a) the x-axis, b) the y-axis?

32. Show that the straight line passing through the points $A(-1, 3)$ and $B(5, 6)$ forms an angle with the x-axis one half the angle formed by the straight line passing through the points $C(-3, -2)$ and $D(0, 2)$.

33. Prove that the line joining the mid-points of two sides of a triangle is parallel to the third side.

34. Show that in a right triangle the median of the hypotenuse is equal in length to half the hypotenuse.

Sec. 2. Straight Line

1. Form the equation of the locus of points equidistant from the points: a) $(0, -4)$ and $(3, 0)$, b) $(-2, 5)$ and $(2, -5)$.

2. Form the equation of the locus of points equidistant from the coordinate origin and from the point: a) $(0, 4)$, b) $(5, 0)$.

3. On the straight line $y = 3x - 2$, find the point of which: a) the abscissa is equal to 3, b) the ordinate is equal to 13.

4. Given the straight lines: $y = 2x - 1$ and $x + y - 2 = 0$. See if they pass through the points $A(1, 1)$, $B(2, 0)$, $C(0, -1)$, $D(-3, 5)$, $E(-2, -5)$ and $O(0, 0)$.

5. Find the equation of a straight line that cuts off, on the axes Ox and Oy, segments equal to a) 3 and 5; b) -7 and 4.

6. Find the equation of a straight line that cuts off a segment of 3 units length on the y-axis and forms an angle with the x-axis equal to: a) $45°$, b) $135°$, c) $180°$.

7. Find the equation of a straight line that cuts off, on the negative side of the y-axis, a segments of 5 units length and is inclined to the x-axis at an angle: a) $30°$, b) $120°$ and c) $0°$.

8. Write the equations of: a) bisectors of coordinate angles, d) straight lines parallel to the x-axis, and y-axis, passing through the point: 1) $(2, -3)$, 2) $(-5, -1)$, 3) $(-3, 0)$, 4) $(0, 4)$.

9. An electric meter registered 2.7 kwhr when the current was switched on. Write the equation of the straight line showing the meter readings under a load of 5 lamps of 60 watts each.

10. A force F is applied at the coordinate origin. Its components on the x-axis and y-axis are, respectively, 4 and -3. Find the equation of the straight line along which the force is directed.

11. What kind of line serves as the graph of uniform motion on the basis of the law $s = vt + s_0$?

12. The graph of uniform motion cuts off segments equal to $-\frac{1}{3}$ and 6 on the axes Ox and Oy. Find the velocity of this motion if unit distance on the x-axis represents one minute and on the y-axis, one metre.

13. A man walking along a horizontal beam resting on two supports, A and B, exerts a pressure on support B that varies with the position of the man on the beam. Find the equation of this pressure versus the distance of the man from the other support A, given the following: weight of beam $P = 120$ kg, its length $l = 5$ m, weight of man $p = 65$ kg.

14. Given the straight lines: a) $5x + 12y - 39 = 0$, b) $4y - 3x + 10 = 0$, c) $x - 2y + 3 = 0$, d) $9x + 12y + 10 = 0$. Without solving the equations for y, find the slopes and the angles formed by the straight lines with the x-axis.

15. Reduce the following equations of straight lines to a form involving the slope:

a) $x - y + 2 = 0$, b) $2x + y - 1 = 0$,
c) $4x - 2y - 1 = 0$, d) $3x + 6y + 2 = 0$,
e) $x - 5y = 0$, f) $2y + 3 = 0$.

Write the values of the slopes and the initial coordinates of these straight lines.

16. Find the segments cut off on the coordinate axes by the straight lines: a) $3x - 2y - 12 = 0$, b) $y = 2 - 3x$.

17. Construct straight lines expressed by the equations in problems 15 and 16.

18. Investigate the positions of the following straight lines with respect to the coordinate axes:

a) $2y - x = 0$, b) $x - y = 0$, c) $x + y = 0$.
d) $x + 1 = 0$, e) $y - 2 = 0$, f) $3x = 0$, g) $4y = 0$.

Draw the straight lines.

19. Reduce the following equations of straight lines to the intercept form:

a) $2x + 3y - 6 = 0$, b) $y = 3 + 6x$, c) $y = x - 1$, and d) $3x - 2y + 5 = 0$.

20. Find the points of intersection of the straight lines:

a) $y = 4 - x$ and $y = 2x + 3$, b) $2x + 3y - 1 = 0$ and $3x - 2y + 5 = 0$, c) $5x + y = 0$ and $10x + 2y - 1 = 0$, d) $x = 0$ and $y = 5$, e) $x = 1$ and $y = 2$, f) $y = -1$ and $x = 2$.

21. The diagonals of a rhombus, equal to 8 and 6 units of length, are respectively treated as the axis of abscissas and the axis of ordinates. Write the equations of the sides of the rhombus.

22. Find the equations of the diagonals of a square, whose side is equal to a, if two of its adjacent sides fall along the coordinate axes and the square as a whole is situated in the third quadrant.

23. A straight line cuts off equal segments on the coordinate axes and passes through the point M (3, 2). Find its equation.

24. A straight line passes through the point (3, 5) in such a way that a segment of it lying between the two axes is divided in half by the point. Find the equation of the line.

25. Find the equation of the straight line passing through the point (3, 2) and forming with the x-axis an angle of a) 45°, b) 135°.

26. Draw straight lines from the point M (6, 2) so that they form an equilateral triangle with the axis Ox.

27. A light ray is directed along the straight line $y = \frac{2}{3}x - 4$; on reaching the x-axis it is reflected. Find the point of incidence of the ray on the axis and the equation of the reflected ray.

28. Find the equation of the straight line passing through the points: a) M_1 (−1, 2) and M_2 (2, 1), b) M_1 (−3, −1) and M_2 (−1, 3), c) M_1 (2, 0) and M_2 (2, −3), d) M_1 (−4, −3) and M_2 (1, −3).

29. Check to see whether the following three points lie on a straight line:

a) (−5, −3), (−1, 1) and (3, 5), b) (−2, 1), (0, 3) and (3, 5).

30. What is the ordinate of the point M whose abscissa is equal to 6 and which lies on a straight line with points: a) A (2, 3) and B (−1, −3), b) A (−6, −1) and B (3, 2)?

31. Find the acute angle between the straight lines:

a) $2x + y - 1 = 0$ and $y - 3x + 1 = 0$,
b) $x\sqrt{3} - y + 2 = 0$ and $x\sqrt{3} + y - 2 = 0$,
c) $x - y + 2 = 0$ and $2x + 3y - 1 = 0$,
d) $2x - y + 3 = 0$ and $4x - 2y - 3 = 0$,
e) $x + 3y + 1 = 0$ and $3x - y - 1 = 0$.

32. Calculate the slopes of the straight lines passing through the points: a) M_1 (1, −1) and M_2 (3, −5), b) M_1 ($\sqrt{3}$, 5) and M_2 ($-\sqrt{3}$, −1), c) M_1 (−3, −2) and M_2 (1, −2).

33. Calculate the interior angles of a triangle with vertices:

a) A (2, 1), B (3, 1) and C (1, 2);
b) A (0, 2), B (2, 0) and C (4, 2).

34. Are there any parallel or perpendicular lines among the straight lines given by the following equations:

a) $x - 3y + 1 = 0$, $2x - 6y + 5 = 0$ and $y = -3x - 2$;
b) $2x - y = 0$, $x + 2y + 3 = 0$ and $y = 2x - 1$?

35. Through the point (3, −6) draw straight lines parallel to the straight lines: a) $y = -2x + 3$, b) $y = 3x$, c) $y = 0$, d) $x = 0$.

36. Draw straight lines through the coordinate origin perpendicular to the straight lines: a) $x - y = 0$, b) $y + 2x - 3 = 0$, c) $x - 1 = 0$, d) $2y + 1 = 0$.

37. Through the point M (1, 2) draw a straight line parallel to the straight line passing through points A (2, −3) and B (3, −1).

38. Through the point M (1, —2) draw a straight line perpendicular to the straight line passing through points A (—3, 2) and B (—1, 3).

39. Form the equations of two lines drawn perpendicular to the straight line $2x - y + 5 = 0$ from the points of its intersection with the axes.

40. Form the equations of straight lines passing through the point (2, 2) and inclined at an angle 45° to the straight line $4x - 5y - 1 = 0$.

41. Draw straight lines passing through the point M (3, 5) and making an angle of 45° with the straight line $x - y + 7 = 0$.

42. Form the equations of the legs of a right-angled isosceles triangle, knowing the vertex of the right angle C (4, —1) and the equation of the hypotenuse $3x - y + 5 = 0$.

43. Find the equation of the straight line which passes through a point that divides (in the ratio of 2 : 3) the segment joining the points A (—2, 3) and B (4, 6), and which is perpendicular to the straight line that cuts off on the x- and y-axes segments —3 and 4, respectively.

44. Find the equation of the straight line which passes through the point N, the segment of which between points A (—3, 2) and B (4, 1) is divided in the ratio 3 : 4, and which is parallel to the straight line on which lie two known points: (2, —1) and (—3, 0).

45. Draw a straight line through the point of intersection of the straight lines $2x + 5y + 8 = 0$ and $3x - 4y - 11 = 0$ so that it: a) is parallel to the straight line $4x - y + 3 = 0$, b) is perpendicular to this line; c) forms an angle of 45° with it.

46. Given points A (—7, 1), B (3, 6), C (5, 3), D (—5, 8). In what ratio are the lines AB and CD divided by their point of intersection?

47. Through the point (2, —3) draw a straight line such that it forms an angle with the x-axis double the angle formed with that axis by the straight line $y = \frac{1}{2}x + 3$.

48. Find the base of the perpendicular drawn from the point (—1, 2) onto the straight line $3x - 5y - 21 = 0$.

49. Find the distance of: a) the point (4, —1) from the straight line $12x - 5y - 27 = 0$, b) the point (2, —3) from the straight line $5x + 12y - 13 = 0$.

50. Find the altitude of a triangle if: a) the vertex of the triangle is the point A (—1, —1) and its base is the straight line $4x - y + 3 = 0$, b) the vertex is (5, —3) and the base is a segment joining the points (0, —1) and (3, 3).

51. Find a point symmetrical to the point Q (—2, —9) about the straight line $2x + 5y - 38 = 0$.

52. Given equations of the bases of a trapezoid: $3x - 4y + 10 = 0$ and $6x - 8y + 15 = 0$. Find its altitude.

53. Given the straight line $3x - 4y - 5 = 0$. Find the equation of a straight line parallel to the given line and separated from it by 2 units.

54. Find the equation of the straight line passing through the point M_0 (x_0, y_0) and parallel to the straight line joining M_1 (x_1, y_1) and M_2 (x_2, y_2).

55. Through the point of intersection of the straight line $7x - 24y - 14 = 0$ with the x-axis draw the bisector of the angle formed by the given straight line with the x-axis.

56. Given two points A $(-3, 8)$ and B $(2, 2)$. Find a point M on the x-axis such that the broken line AMB is of minimum length.

57. At what angle to the x-axis should a ray be sent from the point A $(5, 2)$ so that the ray reflected from the axis should pass through the point B $(-1, 4)$?

58. Form the equation of a straight line passing through the point M_0 (x_0, y_0) and through the point of intersection of two straight lines:

$$A_1x + B_1y + C_1 = 0 \quad \text{and} \quad A_2x + B_2y + C_2 = 0.$$

59. Given equations of the two sides of a parallelogram, $x + y - 1 = 0$ and $3x - y + 4 = 0$, and the point of intersection N $(3, 3)$ of its diagonals. Find the equations of the other two sides of the parallelogram.

60. Establish the boundaries of a square plot of land from the following data:

a) from two poles that represent opposite vertices if the position of the poles is defined by the coordinates A $(2, 1)$ and C $(4, 5)$;

b) from three poles: one at the centre and two at the vertices of one of the sides if the position of the poles is given by the coordinates: of the central pole N $(1, 6)$, of the side poles A $(5, 9)$ and B $(4, 2)$.

61. Determine the position of point M if its distance from point A $(1, -2)$ is equal to 5 units of length and if its direction from point B $(0, -8)$ forms an angle whose tangent is $\frac{1}{2}$ with the positive directions of the x-axis.

62. Three perpendiculars are drawn from the point M $(9, 5)$ to the sides of a triangle whose vertices are $(8, 8)$, $(0, 8)$ and $(4, 0)$. Show that the bases of all these perpendiculars lie on a single straight line and that the point M lies on the circle circumscribed about the given triangle.

63. Check to see that the point of intersection of the altitudes of the triangle lies on a single straight line with the point of intersection of its medians and the centre of the circumscribed circle. Take, for example, the triangle ABC: A $(5, 8)$, B $(-2, 9)$, C $(-4, 5)$.

Sec. 3. Circle

1. Form the equation of a circle having:

a) centre in point $(3, -5)$ and radius equal to 4;

b) centre in point $(-2, 1)$ and passing through the coordinate origin;

c) centre in point $(-3, 0)$ and the end of a diameter in point $(2, -4)$;

2. Form the equation of a circle one of whose diameters is the line segment of the straight line $4x - 3y + 12 = 0$ lying between the coordinate axes.

3. Write the equation of a circle whose centre lies on the axis of abscissas and which passes through the points $M(2, 3)$ and $N(5, -2)$.

4. Write the equation of a circle which passes through the coordinate origin and the point $M(-3, 9)$ and has its centre on the axis of ordinates.

5. Write the equation of a circle that touches the axis of abscissas at the point $A(2, 0)$ and passes through the point $M(-1, 3)$.

6. Form the equations of circles passing through the point $M(2, -1)$ and touching both coordinate axes.

7. Find the normal equations of those circles which pass through points $M(-1, 4)$ and $N(3, 0)$ and have radii equal to 4 units.

8. Find the normal equations of the circles which pass through points $M(4, -2)$ and $N(5, -3)$ and have radii equal to 5 units.

9. Form the equation of a circle that passes through three points:

a) $(0, 0)$, $(7, -7)$, $(8, 0)$;

b) $(0, 4)$, $(1, 2)$, $(3, -2)$.

10. Form the equation of the circumscribed circle of a triangle whose vertices are $A(0, -6)$, $B(1, 1)$ and $C(8, 0)$.

11. Find the centres and the radii of the following circles:

a) $x^2 + y^2 - 6y = 0$;

b) $x^2 + y^2 + 8x - 9 = 0$;

c) $x^2 + y^2 - 10x + 4y + 13 = 0$;

d) $2x^2 + 2y^2 - 6x - 8y - 19 = 0$;

e) $2x^2 + 2y^2 - 5x + 3y - 7 = 0$;

f) $x^2 + y^2 - 12x + 2y + 37 = 0$;

g) $x^2 + y^2 - 2x + 12y + 38 = 0$.

12. Find the equation of a circle of radius 2 units of length and concentric with the circle $x^2 + y^2 + 6x + 8 = 0$.

13. Find the equation of a circle passing through the point $M(-3, 4)$ and concentric with the circle $x^2 + y^2 + 3x - 4y - 1 = 0$.

14. Transform the equation of the circle $x^2 + y^2 + 4x - 12y - 9 = 0$ by shifting the coordinate origin to the centre of the circle, while retaining the directions of the axes.

15. Find the points of intersection, with the axes, of each of the following circles: a) $x^2 + y^2 - 6x - 10y + 9 = 0$, b) $x^2 + y^2 - 10x - 6y + 25 = 0$.

16. What is the position of each of the straight lines: a) $x - 2y - 5 = 0$, b) $3x + 4y + 25 = 0$, c) $x + y - 17 = 0$ relative to the circle $x^2 + y^2 = 25$.

17. Find the centre of a circle of radius $r = 50$, if it is given that the circumference of the circle passes through the point M (0, 8) and cuts off a chord equal to 28 units on the axis of abscissas.

Sec. 4. Ellipse

In the following problems it is required to form the equation of the ellipse if it is known that:

1. The coordinates of the focus F and the point M of the ellipse are: a) F (± 2, 0) and M (2, -3), b) F (± 15, 0) and M (20, 12), c) F (± 22, 0) and M (13, 12).

2. The vertices of the ellipse are the points (± 3, 0) and (0, ± 1).

3. The distance between the foci is equal to 16, and the major axis is 34.

4. The semiminor axis is equal to 4 and the interfocal distance is 15.

5. The semimajor axis is equal to 10 and the eccentricity is 0.6.

6. The eccentricity $e = 0.28$ and the foci have coordinates (± 7, 0).

7. The distances from one of the foci to the ends of the focal axis are, respectively, 18 and 8.

8. The minor axis is equal to 6, the eccentricity is 0.8.

9. The sum of the semi-axes is equal to 8 and the interfocal distance is 8.

10. The eccentricity is equal to $\frac{1}{3}$ and the ellipse passes through the point M (c, 4), where c is the abscissa of a focus.

11. The ellipse passes through points: a) M (6, 4) and N (-8, 3), b) M (6, -6) and N (9, | 6).

12. Calculate the lengths of the axes, the coordinates of the foci and the eccentricity of the ellipse, knowing its equation: a) $25x^2 + 169y^2 = 4225$, b) $3x^2 + 5y^2 = 30$, c) $2x^2 + y^2 = 32$, c) $9x^2 + 25y^2 = 4$, d) $256x^2 + 81y^2 = 576$.

13. Find the equation of the curve obtained by halving all ordinates of the circle $x^2 + y^2 = 36$.

14. Determine the eccentricity of an ellipse if:

a) its minor axis is visible from a focus at a right angle;

b) the distance between the foci is equal to the distance between the ends of the major and minor axes;

c) the ordinate of a point of the ellipse, the abscissa of which is the abscissa of a focus, constitutes $\frac{1}{m}$ th part of the length of the semiminor axis ($m > 1$).

15. Given the eccentricity e of an ellipse, find the relation between its semi-axes.

16. The meridian of the globe has the form of an ellipse with the ratio of the axes $\frac{299}{300}$. Find the eccentricity of the terrestrial meridian.

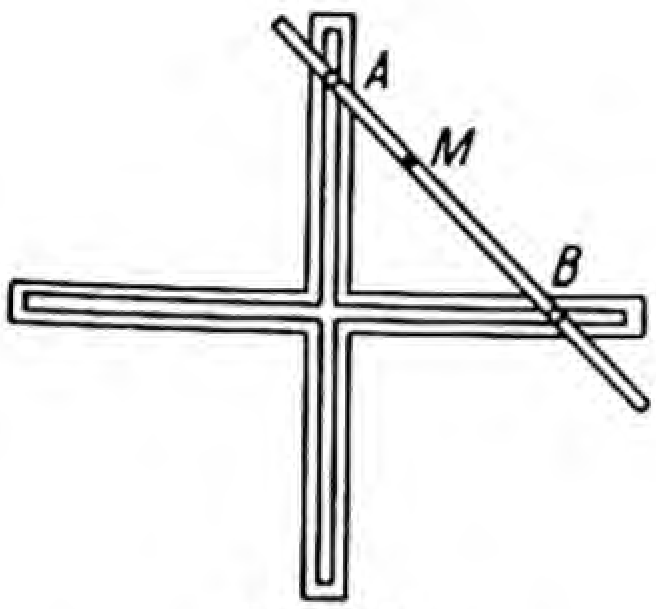

Fig. 155.

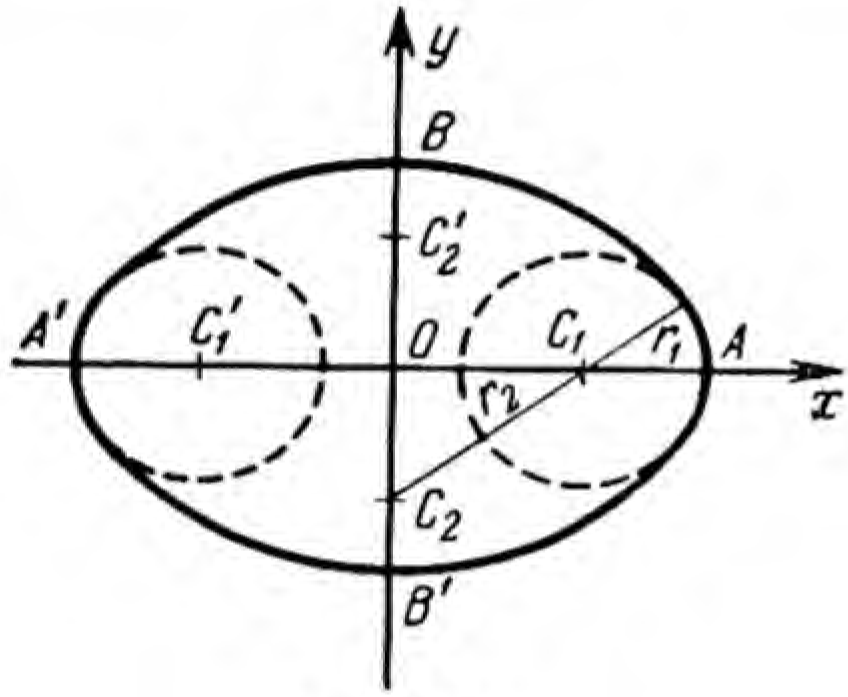

Fig. 156.

17. The earth's orbit is an ellipse with the sun at one of the foci. Knowing the eccentricity of the ellipse, $e = 0.017$, and the semi-axis $a = 150 \cdot 10^6$ km, find out by how much the shortest earth-sun distance (it occurs in December) is shorter than the farthest distance (in June).

18. Find a point on the ellipse $\frac{x^2}{100} + \frac{y^2}{36} = 1$ whose distance from the right focus is four times the distance from the left focus.

19. The ends of a segment AB of constant length slide along the sides of a right angle. Take any point M on the segment and show that the trajectory of the point traced in sliding is an ellipse.

20. Figure 155 shows an elliptical compass in which the length of the sliding line AB and the position of the pencil-holder M can be changed by means of the screws A, B and M. How should one set the compass so as to draw the ellipses:

$$\text{a) } \frac{x^2}{9} + \frac{y^2}{4} = 1, \text{ b) } \frac{x^2}{16} + y^2 = 1, \text{ c) } x^2 + y^2 = 25?$$

21. In practice, approximate drawing of an ellipse by means of a compass is often resorted to in place of exact plotting (Sec. 23); the exact ellipse is replaced by a "dummy" ellipse. From the

point C_1 (Fig. 156), with radius equal to C_1A, and from point C_2, with radius equal to C_2B, draw parts of a circle. The relative values of the radii should be chosen so that both parts of the circle should have a common tangent at the point of connection. Find the relationship connecting the radii $C_1A = r_1$, $C_2B = r_2$ so that the "dummy" ellipse should have the given semi-axes a and b.

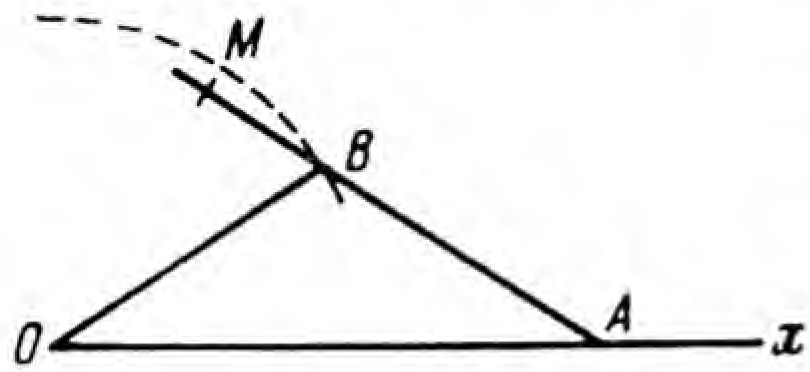

Fig. 157.

22. Figure 157 shows a hinge mechanism in which $OB=BA=a$, $MB = b$. Point O is fixed and B revolves around it along the circumference. Point A moves along the straight line Ox. What curve is described by the point M?

Sec. 5. Hyperbola

The following problems require the formation of the equation of a hyperbola if it is known that:

1. The coordinates of the focus F and the point M of the hyperbola are:

a) $F(\pm 13, 0)$ and $M(22, 12)$,

b) $F(\pm 15, 0)$ and $M(-20, 12)$,

c) $F(\pm 4\frac{1}{2}, 0)$ and $M(10\frac{1}{2}, 8)$.

2. The coordinates of the vertices are $(\pm 1, 0)$ and of the foci, $(\pm 3, 0)$.

3. The real semi-axis is equal to 5 and the vertices divide in half the distance between the centre and focus.

4. The real semi-axis is 6 and the eccentricity $e = 1.5$.

5. The interfocal distance is 26 and the eccentricity is 2.6.

6. The eccentricity of the hyperbola is 1.25 and the hyperbola passes through the point $(2\sqrt{5}, -1.5)$.

7. The hyperbola passes through points $M(-4, 3)$ and $N(\sqrt{10}, -1.5)$.

8. Find the coordinates of the foci, the eccentricity and the equations of the asymptotes of the following hyperbolas: a) $16x^2 - 9y^2 = 576$, b) $3x^2 - 5y^2 = 30$, c) $64x^2 - 225y^2 = 400$.

9. Find the equation of the following hyperbolas:

a) the asymptotes are given by the equations $y = \pm\frac{1}{2}x$ and the foci have coordinates $(\pm\sqrt{10}, 0)$;

b) the asymptotes are given by the equations $y = \pm\frac{3}{4}x$, and the hyperbola passes through the point $(2, 1)$.

10. Determine the angle between the asymptotes if the eccentricity $e = 2$.

11. Calculate the eccentricity of the hyperbola if the angle between the asymptotes is $60°$, $90°$.

12. Find the relation between the eccentricity of the hyperbola and the angle between the asymptote and the principal axis.

13. Find the distance of a point lying on the hyperbola $\frac{x^2}{225} - \frac{y^2}{64} = 1$ from its foci if the abscissa of the point is: a) 15, b) $-16\frac{13}{17}$.

14. Find that point on the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ for which its distance from the left focus is double its distance from the right focus.

15. Find the point $M(x_0, y_0)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that the focal radii vectors are perpendicular to each other.

16. Find the equation of an equilateral hyperbola passing through the point: a) $(4, -2)$, b) $(-3, \sqrt{2})$.

17. Transform the following equations of the hyperbola into asymptotic form: a) $x^2 - y^2 = 12$, b) $x^2 - y^2 = 8$.

18. Transform the following equations of the hyperbola relative to the asymptotes, taking the axis of symmetry of the hyperbola for the coordinate axis: a) $xy = 3$, b) $xy = 5$.

19. Form the equation of the hyperbola having common foci with the ellipse $\frac{x^2}{49} + \frac{y^2}{24} = 1$, assuming the eccentricity of the hyperbola equal to 1.25.

20. Form the equation of the hyperbola whose vertices lie on the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and the foci of the ellipse lie in the foci of the hyperbola.

21. Calculate the length of the side of a square inscribed:

a) in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

b) in the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Investigate into what hyperbolas it is possible to inscribe a square.

22. Find the equations of two perpendiculars dropped from the right focus of the hyperbola $9x^2 - 16y^2 = 144$ onto its asymptotes.

23. Find the points of intersection of the hyperbola $16x^2 - 9y^2 = 144$ with the straight lines: a) $20x + 21y + 12 = 0$, b) $4x - 3y = 0$, c) $y = 2x - 3$.

24. Find the points of intersection of the hyperbola $2x^2 - y^2 = 4$ with the circle $x^2 + y^2 = 8$.

25. Find the points of intersection of the ellipse $\frac{x^2}{49} + \frac{y^2}{24} = 1$ with the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

Sec. 6. Parabola

In each of the following problems it is required to form the equation of the parabola with vertex in the coordinate origin, if it is known that:

1. The coordinates of the focus are: a) (4, 0), b) (0, −3).

2. The equation of the directrix is: a) $x = 1$, b) $y = -2$.

3. The parabola is symmetric about the axis Ox and passes through the point: a) (1, −2), b) (−2, 4).

4. The parabola is symmetric about the axis Oy and passes through the point: a) (−3, 2), b) (2, −3).

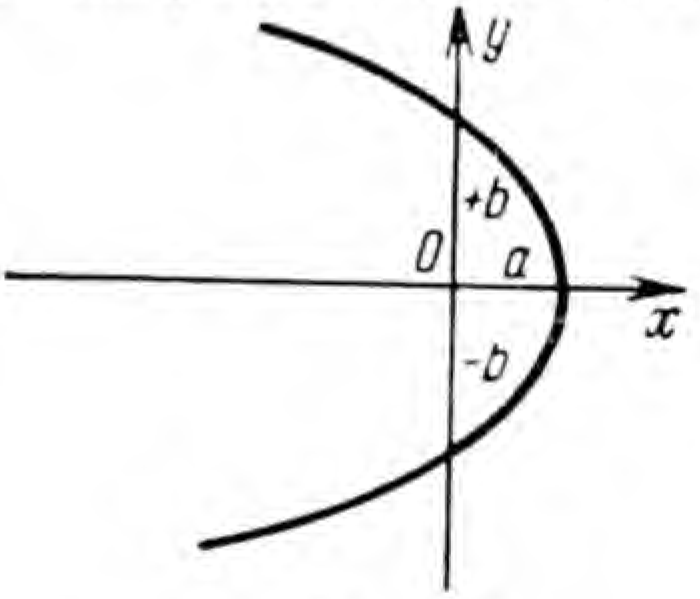

Fig. 158.

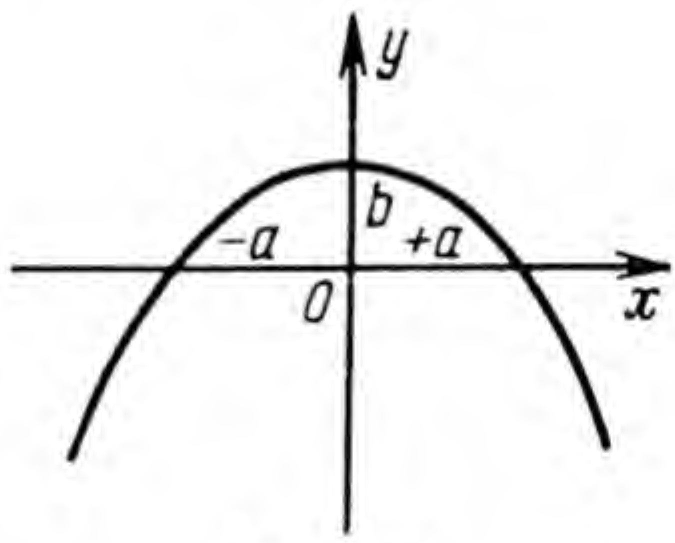

Fig. 159.

5. A stone thrown upwards at an acute angle to the horizon described the arc of a parabola and fell at a distance of 16 metres from the point of throw. Find the parameter of this parabola if the stone reached a maximum height of 12 metres.

6. The mirror of an automobile headlight is parabolic in cross-section. Find the equation of this parabola if we take the diameter of the headlight at 20 cm and the depth at 15 cm. Let the axis of the headlight represent the x-axis and the origin be taken within the mirror.

7. The parabolic mirror of a refractor at an observatory has a focal distance of 20 metres and a diameter of 6 metres. Find the depth of the parabolic cavity which had to be made in sheet glass to make the mirror.

8. Form the equation of a parabola which is:

a) symmetric about the x-axis and cuts off, on this axis, a segment $+a$ and, on the y-axis, segments $\pm b$ (Fig. 158);

b) symmetric about the y-axis and cuts off, on this axis, a segment $+b$ and, on the x-axis, segments $\pm a$ (Fig. 159).

9. The truss shown in Fig. 160 is of parabolic form.

The length of the span $= l$, the sag $= f$; the span is divided into $2n$ equal parts. Determine the length of the vertical (y_1, y_2, . . .) and the diagonal (d_1, d_2, . . . and d_1', d_2', . . .) beams of the truss.

Numerical example: $l = 20$ m, $f = 5$ m and $n = 4$.

10. Solve the same problem with respect to a crescent-shaped truss formed by two parabolas (Fig. 161).

Numerical example: $l = 20$ m, $f = 2$ m (sag of the truss), $f' = 3$ m (flexure of the lower parabola), $n = 4$.

11. Form the equation of the parabola knowing the coordinates of its vertex O' and the focus F: a) O' (2, 3) and F (2, 5); b) O' (3, 0) and F (3, −3); c) O' (1, −2) and F (4, −2); d) O' (2, 0) and F (0, 0).

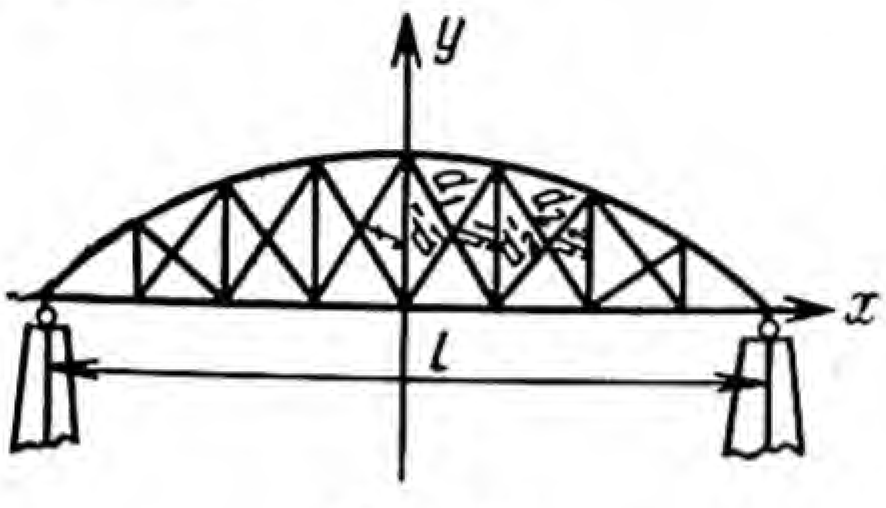

Fig. 160.

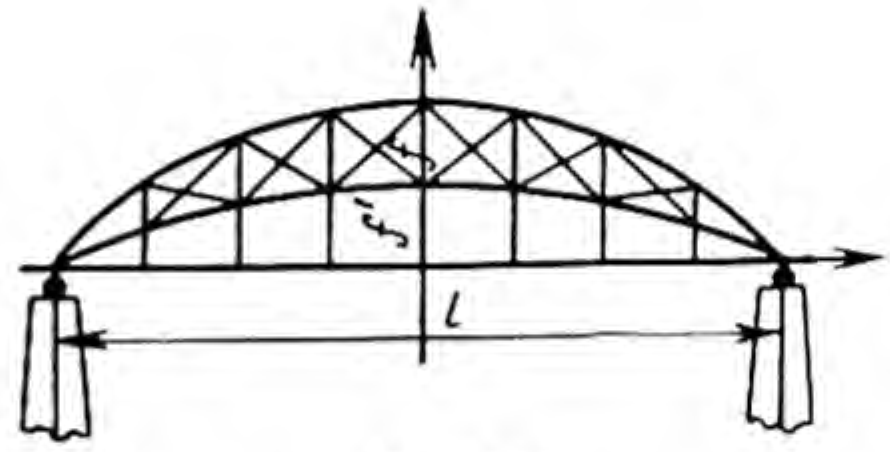

Fig. 161.

12. Form the equation of the parabola knowing the equation of the directrix and the coordinates of the vertex O' of the parabola:

a) $y = -2$ and O' (4, 1), b) $y = 1$ and O' (3, −1), c) $x = 3$ and O' (1, 1), d) $x = 0$ and O' (−1, 0).

13. Form the equation of the parabola knowing the equation of the directrix and the coordinates of the focus F of the parabola:

a) $x = 0$ and F (5, 0),
b) $y = -1$ and F (2, 3),
c) $x = 1$ and F (−2, 2).

14. Form the equation of the parabola knowing that:

a) the axis is parallel to the y-axis, the vertex has the coordinates (2, −1) and the parabola passes through the point (4, 0);

b) the axis is parallel to the x-axis, the coordinates of the vertex are (2, 3) and the parabola passes through the point (1, 1).

15. Determine the coordinates of the vertex, the magnitude of the parameter and the direction of the axis of symmetry in the following parabolas:

a) $y^2 - 10x - 2y + 21 = 0$, b) $y^2 - 6x + 14y + 49 = 0$,
c) $y^2 + 8x - 16 = 0$, d) $x^2 - 6x + 4y - 11 = 0$,
e) $y = x^2 - 8x + 15$, f) $y = x^2 + 6x$,
g) $y = 2x - x^2$, h) $y = x - x^2$.

16. Transform the equations of the following circles into standard form and determine the coordinates of the centre and the magnitudes of the radii:

a) $x^2 + y^2 + 6x - 7 = 0$ and b) $x^2 + y^2 - 10x + 2y + 1 = 0$.

17. Transform the equations of the hyperbolas:

a) $9x^2 - 25y^2 - 18x - 100y - 316 = 0$ and
b) $5x^2 - 6y^2 + 10x - 12y - 31 = 0$

into standard form and determine the coordinates of the centre and the magnitudes of the axes.

18. a) On the parabola $y^2 = 4x$ find a point, the focal radius vector of which is equal to 17.

b) On the parabola $y^2 = 8x$ find a point, the focal radius vector of which is equal to 10.

19. Find the equation of the common chord of the parabola $y^2 = 18x$ and the circle $(x + 6)^2 + y^2 = 100$.

20. Show that the locus of centres of circles passing through a given point and touching a given straight line is a parabola.

Sec 7. Mixed Problems

1. Prove that for every straight line issuing from a point $M(x_0, y_0)$ and intersecting a circle $x^2 + y^2 = a^2$, the product of the distances of the point M from the points of intersection of the line with the circle is the same (the well-known corollary of the theorem in plane geometry about the product of a secant by its external part).

2. The vertex of the right angle of a triangle lies on the straight line $2x + y - 10 = 0$ and the two other vertices, at points $(2, -3)$ and $(4, 1)$. Calculate the area of the triangle.

3. Find the angle between the straight lines passing through the origin and through the points of trisection of the chord $2x + 3y - 12 = 0$ of the parabola $2x^2 - 9y = 0$.

4. The centre of a circle touching the coordinate axis lies on the straight line $3x - 5y + 15 = 0$. Find the equation of the circle.

5. A point with abscissa $1\frac{2}{5}$ is taken in the first quadrant of an ellipse $\frac{x^2}{49} + \frac{y^2}{24} = 1$ and joined with the foci of the ellipse. Prove that the radii vectors thus obtained are mutually perpendicular.

6. The vertex of a parabola lies at the centre of a circle whose radius is equal to $\frac{3}{4}$ the parameter p of the parabola. The chord joining the points of intersection of the two curves serves as one

side of a rectangle inscribed in the circle. Find the length of the sides of the rectangle and the equations of its diagonals.

7. The foci of the ellipses $\frac{x^2}{25}+\frac{y^2}{9}=1$ and $\frac{x^2}{16}+\frac{y^2}{25}=1$ are joined together by straight lines and a circle is inscribed in the rhombus thus obtained. Find the equation of the circle.

8. An ellipse and a hyperbola have common foci separated by the distance $2\sqrt{13}$; the difference of focal semi-axes is equal to 4, and the relation of eccentricities is $\frac{3}{7}$. Form the equations of these curves and find their points of intersection.

9. Given a hyperbola: $x^2 - y^2 - 6x = 0$. Find the equation of the straight line connecting its centre with the centre of a circle which passes through the origin and the points of intersection of a straight line $x - 2y + 4 = 0$ with the coordinate axes.

10. Let the origin O serve as the centre of a circle touching a straight line $3x - y - 10 = 0$ and let O also serve as the vertex of a parabola whose parameter is $\frac{3}{2}$ and whose axis coincides with the x-axis. From the point M — the point of intersection of these two curves in the first quadrant — let a circle be drawn touching the x-axis and intersecting the first circle in points P and Q. Then the point of intersection of the chord PQ with the ordinate of the point M belongs to an ellipse, for which the diameter of the first circle serves as the major axis, and in which the minor axis is equal to the radius of the same circle. Prove this.

Sec. 8. Theory of Limits

1. Show that if x takes on successively the values $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots, \frac{2^n-1}{2^n}, \ldots$, its limit is 1.

2. Show that the limit of $\cos x$ is equal to zero if $x \to \frac{\pi}{2}$.

3. Show that $\lim\limits_{x\to 0} \cos x = 1$.

4. Show that $\lim\limits_{x\to\frac{\pi}{2}} \sin x = 1$.

5. If $x \to 0$, which of the following quantities are infinitesimals: $10x$, x^2, $\sqrt{x}$, ax^2, $\frac{2}{x}$, $\frac{0.001}{x}$, $\frac{x}{x^2}$, $\frac{x^2}{x}$, $x^2 + 0.1x$, $x - x^2$?

6. If x successively assumes the following values: $1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, \ldots$, is x an infinitely large or an infinitely small quantity?

Find the limits of the following expressions:

7. $\frac{10}{x+1}$ when $x \to 4$.

8. $\frac{x^2-3x+6}{x+2}$ when $x \to 2$.

9. $\frac{x^2+6x-1}{5x+3}$ when $x \to 0$.

10. $\frac{x^2-1}{x+1}$ when $x \to 1$.

11. $\frac{x^2+2x-15}{x^2-9}$ when $x \to 3$.

12. $\frac{x^3+2x^2}{x^4-x^3+5x^2}$ when $x \to 0$.

13. $\frac{x^3-3x^2+2x}{x^3-4x^2+3x}$ when $x \to 1$.

14. $\frac{x^3-a^3}{x^2-a^2}$ when $x \to a$.

15. $\frac{x^p-x^q}{x^p+x^q}$ when $x \to 0$ and $p > q$.

16. $\frac{1+x-x^2}{x^2+x-1}$ when $x \to \infty$.

17. $\frac{ax^4+bx^3+cx^2}{kx^4+lx^3+mx^2}$ when $x \to \infty$.

18. $\frac{x^2+5}{2x^3+3}$ when $x \to \infty$.

19. $\frac{ax^2+b}{x^3+c}$ when $x \to \infty$.

20. $\frac{x^2-a^2}{x-a}$ when $x \to \infty$.

21. $\frac{1}{1-x}-\frac{3}{1-x^3}$ when $x \to 1$.

22. $\frac{2x}{x^2-a^2}-\frac{1}{x-a}$ when $x \to a$.

23. $\frac{\sqrt{x}-\sqrt{3}}{x^2-9}$ when $x \to 3$.

24. $\sqrt{x^2+1}-x$, where $x>0$, when $x \to \infty$.

Explain the geometrical significance.

Sec. 9. Functions. Continuity of Functions

1. Setting $f(x)=5+3x-5x^2$, find $f(-1)$.
2. Setting $\varphi(x)=x^2-2$, find $\varphi(x)+4$ and $\varphi(x+4)$.
3. Given: $F(x)=\frac{1}{x^2}+1$; find $F(x-2)$.
4. Given: $F(x)=\frac{2x}{5}+\frac{5}{2x}$; show that $F\left(\frac{5}{2}\right)=-F\left(-\frac{5}{2}\right)$.
5. Given: $f(x)=\frac{1}{x}$; show that $f(x+h)-f(x)=-\frac{h}{x^2+xh}$.
6. Find the values of t for which $f(t)=t^2+t-6$ is equal to zero.
7. An equation with one unknown is written in the form $f(x)=0$. How would you write this equation to express the fact that its roots are 1 and -2?
8. Assuming $f(x)=a^x$, employ this symbol to rewrite the equality $a^x \cdot a^y=a^{x+y}$.
9. Putting $f(x)=\tan x$, rewrite the following formula with the aid of this symbol:

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x \cdot \tan y}.$$

10. Given $f(x)=\sin x$; show that

$$f(x+2z)+f(x)=2\sin(x+z)\cdot\cos z.$$

11. Find the domains of definition of the following functions:

a) $y=\sqrt{4-x^2}$; b) $y=\sqrt{x+1}-\sqrt{3-x}$;

c) $y=\sqrt{3+x}+\sqrt{x-1}$; d) $y=\dfrac{2}{\sqrt{4-x^2}}$.

12. Draw the graphs of the functions:
a) $y=|x|$ on the interval $[-2, +2]$;
b) $y=|x-2|$ on the interval $[-2, +6]$;
c) $y=4x-x^2$ on the interval $[-1, 4]$;

d) $y=\dfrac{1}{x-1}$; e) $y=\dfrac{1}{(x-1)^2}$; f) $y=x+\dfrac{1}{x}$;

g) $y=\dfrac{x^2}{x-1}$; h) $y=\dfrac{1}{1+2^{\frac{1}{x}}}$; i) $y=\dfrac{1}{1+2^{\frac{1}{x-1}}}$;

j) $y=\begin{cases} x+2 & \text{if } -2\leqslant x\leqslant -1; \\ 1 & \text{if } -1\leqslant x\leqslant +1; \\ 2-x & \text{if } +1\leqslant x\leqslant +2. \end{cases}$

13. Find the amount of change in the volume of a cube if its edge x changes by Δx. It is given that $x=2$ metres and $\Delta x=0.1$ metre.

14. Calculate the increment of the function $y=x^3-2x+5$ for an increase in the value of the argument from $x=2$ to $x=2.01$.

15. Find the increment of the function $y=\dfrac{2}{x-1}$ for an arbitrary value of the argument x and an arbitrary increment Δx.

16. Find the increment of the function $y=\log x$ for any positive value of x, given an arbitrary value of the increment Δx.

17. Determine the points of discontinuity of the functions:

a) $y=\dfrac{1}{x}$; b) $y=\dfrac{1}{(x-1)^2}$;

c) $y=\dfrac{1}{x^2-1}$; d) $y=\dfrac{x+7}{x^2+10x+21}$;

e) $y=\dfrac{2x-3}{6x^2-23x+21}$; f) $y=\dfrac{1}{1+2^{\frac{1}{x}}}$.

18. Prove that the following functions are continuous for any real value of $x=c$:

a) $y=\dfrac{2x}{x^2+1}$; b) $y=\sqrt{x}$ $(x\geqslant 0)$;

c) $y=\cos x$; d) $y=a^x$ $(a>0;\ a\neq 1)$;

e) $y=\log_a x$ $(a>1;\ 0<x<+\infty)$.

19. Will a function defined as follows,

$$y=\begin{cases} 2x, \text{ if } 0\leqslant x<1, \\ 3-x, \text{ if } 1\leqslant x\leqslant 2, \end{cases}$$

be continuous on the interval $0\leqslant x\leqslant 2$? Construct the graph.

20. Prove that $\tan x$ is continuous for all values of x except $x=\pm\frac{\pi}{2}+2k\pi$, where k is any integral number.

Sec. 10. Derivative Function

1. Find the ratio $\frac{\Delta y}{\Delta x}$ for the functions:

a) $y=2x^3-x^2+1$ at $x=1$, $\Delta x=0.1$;

b) $y=\frac{1}{x}$ at $x=2$, $\Delta x=0.01$;

c) $y=\sqrt{x}$ at $x=4$, $\Delta x=0.41$.

Show that, as Δx tends to zero, this ratio tends to 4 in the first case, to $-\frac{1}{4}$ in the second case, and to $\frac{1}{4}$ in the third case.

2. The equation of motion of a point is: $s=t^3+10$. Find the velocity of the point at time $t=2$.

3. Angle φ through which a wheel turns during braking is determined by the equation $\varphi=a+bt-ct^2$, where a, b, c are constants and t is the time. Find the angular velocity at time t and determine when the wheel will stop.

4. During the heating of 1 kg of water from 0° to t° the amount of absorbed heat, Q, is given by the formula:

$$Q=t+0.00002t^2+0.0000003t^3 \text{ kcal}.$$

Find the specific heat of water at $t=50$°.

5. A body of mass 2 kg is moving in a straight line according to the law $s=1+t^2$, where s is expressed in cm, t in sec. Determine the kinetic energy $\left(\frac{mv^2}{2}\right)$ of the body after 5 sec.

6. A metal rod of length 50 cm and density d is conical in shape, the diameter of the base of the cone is 1 cm. Find the linear density at the middle point of the rod.

7. Find the slope of the tangent to the parabola $y=x^2$: a) at the coordinate origin, b) at point $x=3$, c) at point $x=-2$, d) at the points of intersection of the parabola with the straight line $y=3x-2$. Show that the tangent at the point $C(x_1, y_1)$ divides the abscissa of this point in half. Find the rule for the construction of the tangent.

8. Find the slope of the tangent to the cubical parabola $y=x^3$: a) at the point (x_1, y_1), b) at the point $x=2$. Can the slope of the

tangent to the cubical parabola $y=x^3$ be negative? Show that a tangent at the point $C(x_1, y_1)$ divides the abscissa of this point in the ratio $2:1$, measuring from the origin. Find the rule for the construction of the tangent.

9. Find the equation of the tangent to an equilateral hyperbola $y=\frac{1}{x}$: a) at the point (x_1, y_1), b) at the point $x=1$, c) at the point $x=-\frac{1}{2}$. Can the slope of the tangent be positive?

10. At what values of the argument are the tangents to the parabolas $y=x^2$ and $y=x^3$ parallel?

11. What is the angle of intersection of the hyperbola $y=\frac{1}{x}$ with the parabola $y=\sqrt{x}$?

12. What are the angles of intersection of the parabolas: $y=\sqrt{2x}$ and $y=\frac{x^2}{2}$?

13. At what point is the normal to the parabola $y=x^2$ perpendicular to the straight line $y=4x+1$?

14. Show that the normals to the curve $y=x^2-x+1$ drawn in points $x_1=0$, $x_2=-1$, $x_3=\frac{5}{2}$ meet at a single point.

Sec. 11. Finding Derivatives

1. $y=3x^2-6x+10$.

2. $y=5x^4+x^2+\frac{1}{3}$.

3. $y=\frac{1}{4}x^4-\frac{1}{2}x^2+x+1$.

4. $y=\frac{1}{5}x^3-\frac{1}{3}x^2-x+0.5$.

5. $y=ax^4+bx^2+c$.

6. $y=ax^3+bx^2-cx+d$.

7. $y=\frac{4x^3}{3}-\frac{2x^5}{5}+\frac{x^7}{7}$.

8. $y=-\frac{7x^6}{8}+\frac{5x^4}{4}-\frac{3x^2}{2}-\frac{2}{3}$.

9. $y=-\frac{ax^4}{b}-\frac{bx^2}{2a}+\frac{a-b}{b}$.

10. $y=\frac{x^3}{a+b}-\frac{x^2}{a-b}-\frac{x}{ab}$.

11. $y=x^a-x^b+3x^{ab}$.

12. $y=x^{2a}-2x^{\log 2}+\pi^2$.

13. $y=x^h-2x^{\pi}+\log 2$.

14. $y=6x^{\frac{7}{2}}+4x^{\frac{5}{2}}+2x^{\frac{3}{2}}$.

15. $y=3x^{-2}-2x^{-\frac{1}{2}}+x$.

16. $y=3\sqrt{x}-4\sqrt[3]{x}+5x\sqrt[5]{x}$.

17. $y=x^2\sqrt{3}+5\sqrt{x^3}+\sqrt{5}$.

18. $y=\frac{1}{x^2}-\frac{7}{x^3}+\frac{1}{3}$.

19. $y=\frac{3a^2}{x^2}-\frac{a}{3x^4}$.

20. $y=2\sqrt{x}-\frac{1}{2\sqrt{x}}$.

For Secs. 84 and 85

Differentiate:

21. $y=(1+5x)(1-7x)$.

22. $y=(1+4x^3)(1-2x^2)$.

23. $y=(2x^2+3x+5)\times$ $\times(5x^2-2x+3)$.

24. $y=(1-x-x^3)(1-x+x^3)$.

25. $y=(ax+b)(cx+d)$.

26. $y=(ax+b)(cx^2+d)$.

27. $y=x(2x-1)(3x+2)$.

28. $y=(x^2-1)(x^2-4)\times$ $\times(x^2-9)$.

29. $y=\frac{5x}{1+x^2}$.

30. $y=\frac{1-x}{1+x}$.

31. $y=\frac{1-x^3}{1+x^3}$.

32. $y=\frac{ax+b}{cx+d}$.

33. $y=\frac{k+ax^2}{l-ax^2}$.

34. $y=\frac{x}{a}+\frac{a}{x}+\frac{x^2}{b^2}+\frac{b^2}{x^2}$.

35. $y=\frac{2x}{5}+\frac{5}{2x}$.

36. $y=\frac{2+x}{3}+\frac{6}{4-x^2}$.

37. $y=3x-\frac{27}{2-x}$.

38. $y=\frac{a^2}{x}+\frac{b^2}{a-x}$.

39. $y=\frac{a}{b+cx^n}$.

40. $z=\frac{1}{t^2-t+1}$.

41. $z=\frac{t-t^3}{\sqrt{\pi}}$.

42. $F(u)=(1+u^3)\left(5-\frac{1}{u^2}\right)$. Find $F'(1)$.

43. $F(u)=\frac{1-\sqrt{u}}{1+\sqrt{u}}$. Find $F'(a)$.

44. $F(u)=(1+\sqrt{u})\cdot\left(1-\frac{1}{\sqrt{u}}\right)$. Find $F'(a)$.

45. $S(t)=\frac{3}{5-t}+\frac{t^2}{5}$. Find $S'(0)$, $S'(2)$.

46. $\varrho=\frac{\varphi}{1+\varphi^2}$. Find $\varrho'(2)$ and $\varrho'(0)$.

For Sec. 88

47. $y=(3x^2+8)^5$.

48. $y=(5-4x^3)^7$.

49. $y=(ax^2+bx+c)^3$.

50. $y=(mx^4+nx^2+p)^4$.

51. $y=(x^m+x^n)^k$.

52. $y=(ax^m+bx^n)^p$.

53. $y=\sqrt{a^2+x^2}$.

54. $y=\sqrt[3]{a+bx+cx^2}$.

55. $y=(ax+b)^2-(ax-b)^2$.

56. $y=\sqrt{5}-7\sqrt{1+x^2}$.

57. $y = x\sqrt{5} - 2\sqrt{3x+5}$.

58. $y = (x^2\sqrt{3} + 5\sqrt{x^3})^4$.

59. $y = \frac{7}{(x^2-1)^3}$.

60. $y = \frac{5}{(3-2x^2)^2}$.

61. $y = \frac{a}{\sqrt{a^2+x^2}}$.

62. $y = \frac{2a}{(1+\sqrt{x})^n}$.

63. $y = \sqrt{(1-x^2)^3} - \sqrt{1-x^2}$.

64. $y = \sqrt[3]{x^2+1} + \sqrt[3]{(x^2+1)^4}$.

65. $y = \sqrt{2+\sqrt{2x}}$.

66. $y = \sqrt{a+\sqrt{ax}}$.

67. $y = (a+x^2)\sqrt{1+bx}$.

68. $y = (2+3x^2)\sqrt{1+5x^2}$.

69. $y = \frac{x^2-2}{(x-1)^4}$.

70. $y = \frac{5x^3}{(5x-4)^3}$.

71. $y = (3x+5)^3(5x+4)^5$.

72. $y = (x+1)(2-3x)^2$.

73. $y = (x+a)^m(x+b)^n$.

$(2x+3)^3$.

74. $y = \frac{(a+x)^m}{(b-x)^n}$.

75. $y = \sqrt{\frac{x+1}{x-1}}$.

76. $y = \sqrt{\frac{x^2-1}{x^2+1}}$.

77. $y = \frac{\sqrt{a^2+x^2}}{x}$.

78. $y = \frac{\sqrt[5]{(1+x^5)^3}}{x^3}$.

79. $y = \frac{x^2}{\sqrt{1+x^3}}$.

80. $y = \frac{2x}{\sqrt[3]{(1+x^2)^2}}$.

81. $y = \frac{x^n}{(1+\sqrt{x})^n}$.

82. $y = \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$.

For Sec. 89

Find the limit of

83. $\frac{\sin kx}{x}$ when $x \to 0$.

84. $\frac{\sin(x-a)}{x-a}$ when $x \to a$.

85. $\frac{\sin 5x}{\sin 3x}$ when $x \to 0$.

86. $x \cdot \cot x$ when $x \to 0$.

For Sec. 90

Find the derivatives of

87. $y = \sin nx$.

88. $y = \sin x^n$.

89. $y = \sin^n x$.

90. $y = \cos(ax)^m$.

91. $y = \cos^m ax$.

92. $y = \cot ax$.

93. $y = \tan^n bx$.

94. $y = 2x + \sin 2x$.

95. $y = \tan\varphi - \varphi$.

96. $y = \tan\frac{\varphi}{2} - \frac{\varphi}{2}$.

97. $\varrho = a\cdot\cos 2\theta$.

98. $\varrho = k\sqrt{\cos 2\theta}$.

99. $x = r(t - \sin t)$.

100. $y = r(1 - \cos t)$.

101. $y = 2\cdot\sin 3x + 3\cos 2x$.

102. $y = A\cdot\sin(\omega t + a)$.

103. $s = \cos\frac{a}{t}$.

104. $s = \sin\frac{1}{t^2}$.

105. $y = \cos^2 2x$.

106. $y = 2\sin^2\frac{x}{2}$.

107. $y = a\cdot\sin^3\frac{1}{x}$.

108. $y = \sin(\cos x)$.

109. $y = \cos(\sin x)$.

110. $y = \cot^3 x + 3\cot x$.

111. $y = \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x$.

112. $y = \frac{2}{3}\tan^3\frac{x}{2} - 2\tan\frac{x}{2} + x$.

113. $y = \sqrt{1 + \sin 2x} - \sqrt{1 - \sin 2x}$.

114. $y = \frac{x}{\sin x}$.

115. $y = \sin x\cdot(\sin x + \cos x)$.

116. $y = \sin x\cdot\sin 2x$.

117. $y = \sin^2 x\cdot(\cos^2 x + 1)$.

118. $f(x) = \frac{1+\cos x}{1-\cos x}$. Find $f'\left(\frac{\pi}{3}\right)$.

119. $f(x) = \frac{1+\sin x}{1-\sin x}$. Find $f'\left(\frac{\pi}{6}\right)$.

120. $f(x) = \frac{a\cdot\sin x}{1+\cos x}$. Find $f'\left(\frac{\pi}{2}\right)$.

121. $f(x) = \frac{1-\cos\frac{x}{3}}{\sin\frac{x}{3}}$. Find $f'(\pi)$.

122. $y = ax\cdot\tan ax$.

123. $y = \sec ax$.

124. $y = \sec^n ax$.

For Sec. 91

125. Knowing $\frac{1}{\log e} = 2.3025851$ and $\frac{1}{\ln 10} = 0.4342945$, find:

a) from the decimal logarithms of the numbers 2, 7, 13 their natural logarithms;

b) $\ln 3 = 1.09861$, $\ln 5 = 1.60944$, $\ln 11 = 2.39790$. Calculate their decimal logarithms.

For Secs. 92-95

Find the derivatives of

126. $y = \log_a (3+x)$.
127. $y = \log_a (1+x^2)$.
128. $y = \ln (a^2 - x^2)$.
129. $y = \ln (5+3x)$.
130. $y = \frac{1}{3} \ln (a^2 + x^2)$.
131. $y = 3 \ln (5-x) + 2 \ln (4+x)$.
132. $y = \ln x^3 + \ln^3 x$.
133. $y = \log_a \sqrt{x}$.
134. $y = \ln 3x + \ln \frac{3}{x}$.
135. $y = \ln \sqrt{x} + \sqrt{\ln x}$.
136. $y = \ln \ln x$.
137. $y = \ln \sqrt{x+1}$.
138. $y = \ln \frac{a+x}{a-x}$.
139. $y = \ln \frac{x^n}{1+x^n}$.
140. $y = \log_a x^2 + \log_a^2 2x$.
141. $y = \log_a^3 2x$.
142. $y = \ln \sqrt[3]{1+x^2}$.
143. $y = \ln \frac{1+\sqrt{x}}{1-\sqrt{x}}$.
144. $y = \ln (x + \ln x)$.
145. $y = a \cdot \ln x^n$.
146. $y = a \cdot \ln^n x$.
147. $y = \ln \sin x + \ln \cos x$.
148. $y = \ln (1 + \cos x)$.
149. $y = \ln \sin^2 x$.
150. $y = \ln \sqrt{\sin 2x}$.
151. $y = \ln \tan \left(\frac{\pi}{4} + \frac{x}{2} \right)$.
152. $y = \sin \ln x$.
153. $y = \tan \ln x$.
154. $y = 5^{x^2 - 2x}$.
155. $y = 2^{x^3 - 3x}$.
156. $y = \frac{1}{2} \cdot 3^{1+x^2}$.
157. $y = x^n + n^x$.
158. $y = e^x + e^{-x}$.
159. $y = e^{\frac{x}{2} + \frac{2}{x}}$.
160. $\rho = a^{\varphi}$.
161. $\rho = a^{\ln \varphi}$.
162. $y = a \cdot e^{\sqrt{x}}$.
163. $y = e \cdot a^{\frac{x}{e}}$.
164. $y = 5^{\sin^2 x}$.
165. $y = a^{\tan nx}$.
166. $y = e^{\tan \frac{1}{x}}$.
167. $y = a^{e^x}$.
168. $y = e^{a^x}$.
169. $y = (x-3) e^{2x} - 4x \cdot e^x + 3$.
170. $y = x^2 \cdot e^x \cdot \cos x$.
171. $\frac{\ln x}{x}$.
172. $y = x \cdot \ln x - x$.
173. $y = e^{x \cdot \ln x}$.

174. $y=\frac{e^x-e^{-x}}{e^x+e^{-x}}$.

175. $y=\frac{a^x-a^{-x}}{a^x+a^{-x}}$.

176. $y=x\cdot\cot x-$
$-\ln\sin x+\frac{x^2}{2}$.

177. $y=\frac{1}{2}x^2\ln\tan x=\sin x$.

178. $f(x)=\ln\tan\frac{x}{2}-\frac{\cos x}{\sin^2 x}$. Find $f'\left(\frac{\pi}{4}\right)$.

179. $f(x)=e^{\pi x}\sin\pi x$. Find $f'\left(\frac{1}{2}\right)$.

180. $y=e^{ax}(\sin ax-\cos ax)$.

181. $f(x)=\ln\frac{1+\tan x}{1-\tan x}$. Find $f'\left(\frac{\pi}{6}\right)$.

182. $f(x)=\sqrt[3]{\ln x}$. Find $f'(e)$

183. $f(x)=x\sqrt[3]{\ln x}$. Find $f'(e)$.

184. $y=\ln(\ln^2 x)$.

185. $f(x)=\frac{x}{e^{\sqrt{x}}}$. Find $f'(0)$.

186. $y=\ln e^{7x^2-x+1}$.

187. $y=\ln\left(x+\sqrt{1+x^2}\right)$.

188. $y=\ln\left(\sqrt{x+a}+\sqrt{x}\right)$.

189. $y=\ln\frac{x}{\sqrt{x^2-1}-x}$.

For Sec. 96

Find the derivatives of

190. $y=\arcsin\frac{x}{a}$.

191. $y=\arctan\frac{1}{x}$.

192. $y=\text{arc cosec}\, x$.

193. $y=e^{\arcsin x}$.

194. $y=\arcsin(\cos x)$.

195. $y=\arccos(\sin x)$.

196. $y=\arctan(\ln x)$.

197. $y=\text{arc cot}\left(\ln\frac{1}{x}\right)$.

198. $y=\ln\arctan\left(\tan\frac{x}{2}\right)$.

199. $y=\arctan(\ln\sin x)$.

200. $y=\arctan\frac{a+x}{a-x}$.

201. $y=x\cdot\arcsin x$.

202. $y=\frac{1}{2}x^2\arctan x+$
$+\frac{1}{2}\arctan x-\frac{1}{2}x$.

203. $y=\sqrt{a^2-x^2}+a\cdot\arcsin\frac{x}{a}$.

204. $y=x\sqrt{1-x^2}+\arcsin x$.

205. $y=\arctan\frac{a}{x}+\ln\sqrt{\frac{x-a}{x+a}}$.

For Sec. 97

Find the derivatives of

206. $y = x^{\frac{1}{x}}$.

207. $y = (\sin x)^x$.

208. $y = (\sin x)^{\sin x}$.

209. $y = x^{\ln x}$.

210. $y = x^{\frac{1}{\ln x}}$.

211. $y = x^{\arcsin x}$.

212. $y = e^{x^x}$.

213. $y = \left(\frac{x}{n}\right)^{nx}$.

For Sec. 98

Find the derivatives of order indicated by the answer

214. $y = \arcsin x$. Ans. $y'' = \frac{x}{\sqrt{(1-x^2)^3}}$.

215. $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$. Ans. $y'' = \frac{1}{2a}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$.

216. $y = \frac{a \cdot \sin x}{1 + \cos x}$. Ans. $y'' = \frac{a \cdot \sin x}{(1 + \cos x) \cdot {}^2}$.

217. $y = \ln \sin x$. Ans. $y''' = 2 \cot x \cdot \operatorname{cosec}^2 x$.

Find the general expressions for derivatives of the nth order

218. $y = x^m$. Ans. $y^{(n)} = m(m-1)\ldots$
$\ldots[m-(n-1)]\, x^{m-n}$.

219. $y = \frac{1}{x+a}$. Ans. $y^{(n)} = (-1)^n \cdot n!$
$(x+a)^{-(n+1)}$.

220. $y = e^{a+bx}$. Ans. $y^{(n)} = b^n \cdot e^{a+bx}$.

221. $y = a^{bx}$. Ans. $y^{(n)} = a^{bx} \cdot b^n \cdot \ln^n a$.

222. $y = \cos x$. Ans. $y^{(n)} = \cos\left(x + n \cdot \frac{\pi}{2}\right)$.

223. $y = \sin x$. Ans. $y^{(n)} = \sin\left(x + n \cdot \frac{\pi}{2}\right)$.

224. $y = \ln(ax + b)$. Ans. $y^{(n)} = \frac{(-1)^{n-1} \cdot (n-1)!\, a^n}{(ax+b)^n}$.

Verify the equalities

225. If $y = \sin x - x \cdot \cos x$, then $y'' + y = 2 \sin x$.

226. If $y = a \cdot \cos 2x + b \cdot \sin 2x$, then $y'' + 4y = 0$.

Sec. 12. Studying Functions with the Help of Derivatives. Maximum and Minimum. Velocity and Acceleration

Define the intervals of variation of x in which the following functions decrease and increase:

1. $y=x^2-4x+6$.
2. $y=2x^2-4x+5$.
3. $y=2x^3-3x^2+1$.
4. $y=x^3-3x^2+5$.
5. $y=x^4-2x^2+1$.
6. $y=x^4-4x^2+5$.

Test the following functions for maxima and minima:

7. $y=2x^2-5x-3$.
8. $y=4+3x-x^2$.
9. $y=4-3x-x^2$.
10. $y=x^3-27x$.
11. $y=x^2-2x^3$.
12. $y=x^3+3x-7$.
13. $y=x^3+9x-1$.
14. $y=x^3+x^2-8x+1$.
15. $y=x^3-2x^2-7x+2$.
16. $y=x^4-4x^3+4x^2-15$.
17. $y=x^4-14x^2+24x-3$.
18. $y=3x^4-4x^3-6x^2+12x-8$.
19. $y=(x-1)(x+1)^3$.
20. $y=x(x-1)^2\cdot(x+1)^3$.
21. $y=\frac{2x}{5}+\frac{5}{2x}$.
22. $y=3x-\frac{27}{2-x}$.
23. $y=\frac{x}{x^2+1}$.
24. $y=\frac{1-x}{x^2-x+4}$.
25. $y=\frac{3x^2+5x+25}{2+x}$.
26. $y=\frac{x^2+3x-10}{x+5}$.
27. $y=\frac{1-x}{(1+x)^2}$.
28. $y=2+\sqrt[3]{(x-2)^2}$.
29. $y=3-\sqrt[3]{(x+4)^2}$.
30. $y=e^{-x^2}$.
31. $y=x^2\cdot e^{-x}$.
32. $y=\frac{x}{\ln x}$.

In the following examples find the maxima and the minima of the functions in the range of the simplest values of the arcs.

33. $y = \sin x + \cos x$. 34. $y = \tan x + \cot x$.

35. $y = x + \tan x$. 36. $y = \sin x\,(1 + \cos x)$.

37. $y = \frac{\sin x}{1+\tan x}$.

38. Divide 10 into two parts such that their product yields the maximum.

39. Divide 10 into two parts such that the sum of twice one part and the square of the other is a minimum.

40. Find the positive number which when added to its reciprocal will yield a minimum sum.

41. Find the positive number which differs from its square by a maximum value.

42. Of all rectangles with perimeter $2p$, find that one with the greatest area.

43. Of all the rectangles inscribed in a circle of radius R, find that one which has: a) the largest area, b) the greatest perimeter.

44. Of all the isosceles triangles inscribed in a circle of radius R, find that one which has: a) the largest area, b) the greatest perimeter.

45. Which of the right-angled triangles with hypotenuse c has: a) the greatest sum of the legs, b) the largest area?

46. A rectangular window has a semicircular top. The entire perimeter of the window equals $2p$. What dimensions should the window have so as to yield the largest possible area?

47. What should be the radius of a circle so that its sector having a perimeter $2a$ should have the largest possible area?

48. What should be the relation between the radius and the altitude of a cylinder for it to have a given volume v and the smallest total surface?

49. What radius and altitude should be chosen for a conic tent so that it would require the least possible amount of material to enclose a given volume v?

50. Find the sides of a rectangle of maximum area inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

51. Determine the width of a rectangle of maximum area inscribed in a segment of altitude h of the parabola $y^2 = 2px$.

52. A derrick is set up in a field 9 km away from the nearest point of a highway. It is necessary to send a messenger from the derrick to a village on the highway 15 km away from the aforesaid point (the highway is considered rectilinear). If the messenger can do 8 km an hour over the fields and 10 km an hour on the highway, to what point on the highway should he go so as to reach the village in the shortest possible time?

53. Ship A, at a distance of 75 miles to the east of another ship, B, is moving westwards at a speed of 12 miles an hour; ship B is moving northwards at 9 miles an hour. When will the ships be closest to each other?

54. We have a table-lamp that can be lowered or raised; to what height should it be raised to give the brightest illumination to a book lying on the table a cm away from the centre of the base of the lamp? The brightness of illumination is directly proportional to the sine of the angle of inclination of the light to the plane of the book and inversely proportional to the square of the distance of the book from the source of light $\left(I = \frac{k \cdot \sin\varphi}{r^2}\right)$.

55. Let a quantity x be measured n times with equal care. The results of measurements $l_1, l_2, l_3, \ldots, l_n$ will differ from each other, though slightly. In each measurement the difference $|x - l_k|$ is the absolute error. According to the theory of errors ("the method of least squares") the most probable value of the quantity will be the one at which the sum of the squares of the errors:

$$(x - l_1)^2 + (x - l_2)^2 + (x - l_3)^2 + \ldots + (x - l_n)^2$$

reaches a minimum. Show that the most probable value is the arithmetical mean of the results of all the measurements:

$$x = \frac{l_1 + l_2 + \ldots + l_n}{n}.$$

Investigate the following functions and construct their graphs:

56. $y = x^4$.
57. $y = x^3$.
58. $y = 2x^3 - 3x^2 + 1$.
59. $y = x^2 - 2x^3$.
60. $y = e^{-x^2}$.
61. $y = \frac{2x}{5} + \frac{5}{2x}$.
62. $y = \frac{3x^2 + 5x + 25}{2 + x}$.
63. $y = \frac{x^2}{2} + \frac{1}{x}$.
64. $y = x + 2\sqrt{4 - x}$.

65. A point is moving in a straight line according to the law $s = t^3 - 4t^2 + 10t + 1$. Determine its velocity and acceleration at $t = 0$, $t = 1$, $t = 2$.

When does the point have minimum velocity and what is the magnitude of this velocity?

66. A point is in rectilinear motion according to the law $s = a \cdot e^{-kt}$. Determine the initial velocity and initial acceleration.

67. A point is in rectilinear motion according to the law $s = \frac{v_0}{2}(e^t - e^{-t})$.

Show that the acceleration $a = s$.

68. A point is in rectilinear motion according to the law:

$$s = a \cdot \tan \omega t.$$

Find the velocity and acceleration.

Sec. 13. Differential

1. Show that the differential of a linear function is equal to the increment of the function.

2. Calculate the increment and the differential of the function $y = x^3 - 2x + 5$ on a change in the values of the argument from $x = 2$ to $x + \Delta x = 2.01$.

3. Calculate the increment and the differential of the function $y = \frac{2}{x-1}$ on a change in the argument from 3 to 3.001.

Find the differentials of the following functions:

4. $y = ax^3 + bx^2 - cx + d.$ 5. $y = \sqrt{x} + \frac{1}{\sqrt[3]{x}}.$

6. $y = \frac{x}{c} - \frac{c}{x}.$ 7. $y = \sqrt[3]{1 + x^2}.$

8. $y = \frac{x}{\sqrt{x^2+16}}$ on a change in x from 3 to 3.2.

9. $y = \sqrt{\frac{4-x}{1+x}}$ on a change in x from 0 to 0.1.

10. $y = \frac{\sin x + \cos x}{\sin x - \cos x}.$

11. $y = \frac{1}{3}\tan^3 x - \tan x + x.$ 12. $y = \ln x^2 + \ln \sqrt{x}.$

13. $y = x \cdot \ln x.$ 14. $y = \ln \sqrt{1+x^2}.$

15. $y = \arcsin \frac{x+1}{\sqrt{2}}.$ 16. $y = e^x (\sin x + \cos x).$

17. For a measurement accurate to 0.1 m it is found that the side of a square is equal to 5.2 m. Determine the maximum errors—absolute and relative—for the area of the square.

18. For a measurement accurate to 0.01 m it is found that the edge of a cube is equal to 1.05 m. Find the maximum errors—absolute and relative—for the volume of the cube.

19. The period of oscillation of a pendulum is defined by the formula $T = 2\pi \sqrt{\frac{l}{g}}$, where l is the length of the pendulum, and g is the acceleration of gravity. In measuring l an error Δl was committed. Determine the relative error in the calculated value of T.

20. Show that on raising a base to the nth power the relative error in the former increases n times, and on extracting the nth root the relative error in the radicand decreases n times.

Sec. 14. Indefinite Integral

For Secs. 116-119

1. $\int x^3\,dx.$ 2. $\int \sqrt[3]{x}\,dx.$

3. $\int \frac{dx}{\sqrt{x}}.$ 4. $\int \frac{dx}{x\sqrt{x}}.$

5. $\int (x^4 - 3x^3 + 5x^2)\,dx.$ 6. $\int (1-2x)(1+3x)\,dx.$

7. $\int \left(x^2 - \frac{1}{x^2}\right) dx.$

8. $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx.$

9. $\int \frac{2x+3}{x}\, dx.$

10. $\int \frac{(x+2)^2}{x}\, dx.$

11. The velocity of a body starting from rest is $v_0 + at$ after t seconds. Find the distance travelled in t seconds.

12. Find the equation of a curve passing through the point (1, 1) and having a slope of the tangent to any point equal to $3x - 1$.

For. Sec. 120

13. $\int \left(\sqrt{2x} - \frac{1}{\sqrt{2x}}\right) dx.$

14. $\int \frac{dx}{1-x}.$

15. $\int \frac{dx}{(x+1)^2}.$

16. $\int \frac{4x^3\, dx}{(x^4+a^4)^2}.$

17. $\int \frac{x\, dx}{(x^2+a^2)^5}.$

18. $\int \frac{ax\, dx}{a^2+x^2}.$

19. $\int \frac{x^{-\frac{1}{2}}\, dx}{1+\sqrt{x}}.$

20. $\int \frac{dx}{x-\sqrt{x}}.$

21. $\int \frac{\ln x \cdot dx}{x}.$

22. $\int \frac{dx}{x\sqrt{\ln x}}.$

23. $\int \frac{dx}{x \cdot \ln x^2}.$

24. $\int e^{-4x} \cdot dx.$

25. $\int \frac{dx}{3 \cdot e^x}.$

26. $\int 12a^{2x} \cdot \ln a \cdot dx.$

27. $\int 2^{4x^2-8}\, x\, dx.$

28. $\int \frac{x^3+7x^2-4x-3}{x-1}\, dx.$

29. $\int \frac{x^3-x^2+x+4}{x+1}\, dx.$

30. $\int x\sqrt{a^2-x^2}\, dx.$

31. $\int x^2\sqrt{x^3-1} \cdot dx.$

32. $\int \frac{2x+a}{x^2+ax+b}\, dx.$

33. $\int \frac{(2x+a)\, dx}{\sqrt{x^2+ax+b}}$

34. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot dx.$

35. $\int \frac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{e^{\frac{x}{a}} - e^{-\frac{x}{a}}}\, dx.$

36. $\int 3 \cdot e^x \cdot \sqrt{1+e^x} \cdot dx.$

37. $\int 2x\sqrt{2x-1}\, dx.$

38. $\int \frac{dx}{1+e^x}.$

39. $\int \frac{dx}{\sqrt{x+a}-\sqrt{x}}$.

40. $\int \frac{dx}{\sqrt{x}+\sqrt{x+1}}$.

41. $\int \frac{\sqrt{x^3}-1}{\sqrt{x}-1}\,dx$.

42. $\int \frac{x-1}{\sqrt[3]{x-1}}\,dx$.

43. $\int \frac{dx}{\arcsin x\cdot\sqrt{1-x^2}}$.

44. $\int \frac{\arctan x\cdot dy}{1+x^2}$.

45. $\int \sqrt{\frac{x-1}{x}}\cdot\frac{dx}{x^2}$.

46. $\int \frac{dx}{\left(\ln\frac{x^2}{1+x^2}\right)\cdot(x+x^3)}$.

47. The velocity v of a body is given by the equation $v=3t^2+2t$; the path covered in $t=2$ sec is equal to 12 m. Find the law of motion.

48. Find the law of rectilinear motion, knowing that at every instant t the acceleration $a=t^2\,\text{m/sec}^2$ and that the body started from rest with an initial velocity $v=2$ m/sec.

49. Find the equation of a curve if the tangent to any point forms an angle with the axis Ox, the tangent of which is equal to $\frac{1}{2}\left(e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right)$, and the curve passes through the point $M(0, a)$.

For Sec. 121

50. $\int \sin(nx+a)\cdot dx$.

51. $\int \cos\frac{2x-3}{5}\,dx$.

52. $\int (\sin 2x-\cos 3x)\cdot dx$.

53. $\int \left(\cos ax+\sin\frac{x}{a}\right)dx$.

54. $\int \left(\cos\frac{x}{a}-\sin ax\right)\cdot dx$.

55. $\int (\sec^2 x+\operatorname{cosec}^2 x)\,dx$.

56. $\int (\sec^2 5x-\operatorname{cosec}^2 5x)\,dx$.

57. $\boxed{\int \tan x\,dx=-\ln\cos x+c}$

58. $\boxed{\int \cot x\,dx=\ln\sin x+c.}$

59. $\int \tan x\cdot\sec x\,dx$.

60. $\int \cot x\cdot\operatorname{cosec} x\cdot dx$.

61. $\int \sqrt{\tan x}\cdot\frac{dx}{\cos^2 x}$.

62. $\int \frac{dx}{\sqrt{\cot x}\sin^2 x}$

63. $\int \sin^2 x\cdot\cos x\,dx$.

64. $\int \cos^2\frac{x}{2}\cdot\sin\frac{x}{2}\cdot dx$.

65. $\int \frac{\cos x\,dx}{\sin^2 x}$.

66. $\int \frac{\sin ax\,dx}{\cos^3 ax}$.

67. $\int x^2\cdot\sin(2x^3+3)\cdot dx$.

68. $\int \cos\frac{3}{x}\cdot\frac{dx}{x^2}$.

69. $\int \frac{\tan x}{\sin 2x}\,dx$.

70. $\int \frac{\cos x \cdot dx}{1+2\sin x}$.

71. $\int \frac{\sin x\, dx}{a-b\cdot\cos x}$.

72. $\int \frac{1+\cos x}{(x+\sin x)^3}\, dx$.

73. $\int \frac{4dx}{1-\cos x}$.

74. $\int \frac{dx}{a(1+\cos x)}$.

75. $\int \tan^2 x\cdot dx$.

76. $\int \cot^2 x\, dx$.

77. $\int e^{\cos x}\cdot\sin x\, dx$.

78. $\int \cos x\cdot a^{\sin x}\, dx$.

79. $\int \frac{1+\sin x}{\cos^2 x}\, dx$.

80. $\int \frac{1-\cos 2x}{\sin^2 2x}\, dx$.

81. $\int \frac{\sin x\cdot\cos x\cdot dx}{\sqrt{\cos^2 x-\sin^2 x}}$.

82. $\int \frac{\sin x\cdot\cos x\cdot dx}{\sqrt{2-\sin^2 x}}$.

83. $\int \frac{dx}{1-\sin x}$.

84. $\int \sqrt{1+\sin x}\cdot dx$.

85. $\int \frac{dx}{\sin x\cdot\cos x}$.

86. $\int \frac{dx}{\sin^2 x\cdot\cos^2 x}$.

87. $\int \frac{\sin x-\cos x}{\cos^3 x}\, dx$.

88. $\int \frac{\cos x+\sin x}{\sin^3 x}\, dx$.

89. $\int \frac{\cos x+2\sin x}{1+\sin x-2\cos x}\, dx$.

90. $\int \frac{x\cdot\cos x\cdot dx}{(x\cdot\sin x+\cos x-1)^m}$.

91. Find the function whose derivative is equal to $\sin x+\cos x$ and its value at $x=\pi$ is equal to 1.

92. At any time the velocity of a body is $v=\cos 2t$. Find the law of motion, knowing that in time $t=\frac{\pi}{2}$ sec the body covers a distance s, equal to 6 metres.

In the following examples the values of the letters are positive. The integrals in rectangles can be used as formulas. The examples with asterisks should be omitted if formulas (XI) and (XII) are not studied.

Find:

93. $\int \frac{dx}{9+4x^2}$.

94. $\int \frac{dx}{4+9x^2}$.

95. $\int \frac{dx}{7+5x^2}$.

96. $\int \frac{dx}{2+3x^2}$.

97*. $\int \frac{dx}{4-9x^2}$.

98*. $\int \frac{dx}{7-5x^2}$.

99. $\int \frac{dx}{\sqrt{9-x^2}}$.

100. $\int \frac{dx}{\sqrt{4-x^2}}$.

101. $\int \frac{dx}{\sqrt{5-7x^2}}$.

102. $\int \frac{dx}{\sqrt{7-5x^2}}$.

103*. $\int \frac{dx}{\sqrt{x^2 \pm 9}}$.

104*. $\int \frac{dx}{\sqrt{x^2 \pm 4}}$.

105*. $\int \frac{dx}{\sqrt{4x^2 - 3}}$.

106*. $\int \frac{dx}{\sqrt{1 + 7x^2}}$.

107. $\int \frac{5x\,dx}{\sqrt{1 - x^4}}$.

108. $\int \frac{x^2\,dx}{\sqrt{a^2 - x^6}}$.

109. $\int \frac{dx}{e^{-x} + e^x}$

110. $\int \frac{e^x\,dx}{\sqrt{1 - e^{2x}}}$.

111. $\int \frac{\cos x \cdot dx}{a^2 + \sin^2 x}$.

112*. $\int \frac{\sin x\,dx}{a^2 - \cos^2 x}$.

113. $\int \sqrt{\frac{1-x}{1+x}}\,dx$.

114. $\int \sqrt{\frac{a+x}{a-x}}\,dx$.

115. $$\boxed{\int \frac{dx}{a + bx^2} = \frac{1}{\sqrt{ab}} \cdot \arctan \sqrt{\frac{b}{a}}\,x + c}$$

116*. $$\boxed{\int \frac{dx}{a - bx^2} = \frac{1}{2\sqrt{ab}} \ln \frac{\sqrt{a} + x\sqrt{b}}{\sqrt{a} - x\sqrt{b}} + c}$$

117. $$\boxed{\int \frac{dx}{\sqrt{a - bx^2}} = \frac{1}{\sqrt{b}} \arcsin \sqrt{\frac{b}{a}}\,x + c}$$

118*. $$\boxed{\int \frac{dx}{\sqrt{a + bx^2}} = \frac{1}{\sqrt{b}} \ln \left(x\sqrt{b} + \sqrt{a + bx^2}\right) + c}$$

119*. $$\boxed{\int \frac{dx}{\sqrt{ax^2 - b}} = \frac{1}{\sqrt{a}} \ln \left(x\sqrt{a} + \sqrt{ax^2 - b}\right) + c}$$

120. $\int \frac{dx}{x\sqrt{1 - \ln^2 x}}$.

121. $\int \frac{dx}{a\cos^2 x + b\sin^2 x}$.

122*. $\int \frac{dx}{a\cos^2 x - b \cdot \sin^2 x}$.

For Sec. 122

123. $\int \cos^3 x\,dx$.

124. $\int \sin^5 x \cdot dx$.

125. $\int (\cos^2 x - \sin^2 x) \cdot \sin x\,dx$.

126. $\int \sin^4 x \cdot \cos^3 x\,dx$.

127. $\int \cos^3 x \cdot \sin 2x\,dx$.

128. $\int \frac{\cos^3 x\,dx}{1 - \sin x}$.

129. $\int \frac{\sin^3 x \cdot dx}{1+\cos x}$.

130. $\int \cos^2(ax)\cdot dx$.

131. $\int \sin^2(ax)\,dx$.

132. $\int \sin^4 x\cdot dx$.

133. $\int \sin^2 x\cdot\cos^4 x\cdot dx$.

134. $\int \cos^2 x\cdot\cos 2x\cdot dx$.

135. $\int \cot^3 x\,dx$.

136. $\int \tan^4 x\cdot dx$.

137. $\int \tan^5 x\,dx$.

Sec. 15. Definite Integral

Evaluate:

1. $\int\limits_0^3 5x^2\,dx$.

2. $\int\limits_{-1}^{+2} (2x+3x^2+4x^3)\,dx$.

3. $\int\limits_2^3 \frac{dx}{x^2}$.

4. $\int\limits_1^4 \frac{32\,dx}{x^3}$.

5. $\int\limits_1^3 \sqrt{x}\,dx$.

6. $\int\limits_1^4 \frac{dx}{\sqrt{x}}$.

7. $\int\limits_1^3 \sqrt{2x-2}\,dx$.

8. $\int\limits_1^3 \frac{x\,dx}{1+x^2}$.

9. $\int\limits_0^1 e^x\,dx$.

10. $\int\limits_{-1}^{+1} a^x\,dx$.

11. $\int\limits_0^1 \frac{dx}{1+x^2}$.

12. $\int\limits_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$.

13. $\int\limits_0^{\frac{\pi}{6}} \sin 3x\,dx$.

14. $\int\limits_0^{\frac{\pi}{2}} \sin^2 x\,dx$.

15. $\int\limits_0^{\frac{\pi}{2}} \cos^2 x\,dx$.

16. $\int\limits_0^{\pi} \sin^3 x\cdot\cos^2 x\,dx$.

17. The velocity of a body moving in a straight line is expressed by the formula $v=2+t$. Find the distance covered between $t=2$ and $t=5$.

18. A body is thrown vertically downwards with an initial velocity of 10 m/sec. Knowing that $\frac{dv}{dt}=g$, find the distance covered by the body in t seconds.

19. Find the definite integral of the function $4x^3$, the antiderivative of which is zero at $x=2$.

20. Find the definite integral of the function $\frac{x^2}{1+x^3}$, if its antiderivative is zero at $x=1$.

21. Find the definite integral of the function $\tan x$, if its antiderivative is zero at $x=0$.

22. Find the definite integral of the function $\sin 2x$, if its antiderivative is zero at $x=\frac{\pi}{4}$.

23. Calculate the area of half a sinusoidal wave $y=\sin 2x$.

24. Calculate the area enclosed by an equilateral hyperbola $xy=1$ and the x-axis between the ordinates $x=1$, and $x=a\ (a>1)$.

Calculate by integration the areas bounded by the following lines:

25. $y=4x,\ y=0,\ x=3$.

26. $y=2x+1,\ y=0,\ x=3$.

27. $2y-3x-5=0,\ y=0,\ x=1,\ x=3$.

28. $x+2y+8=0,\ y=0,\ x=-4$.

29. $y=2x-x^2,\ y=0$.

30. $y=x^2-x,\ y=0$.

31. $y=x^3,\ y=2x$.

32. $y^2=4x,\ y^2=x^3$.

33. $y^2=4(x+1),\ y=x+1$.

34. $y^2=2(x+4),\ y=x+4$.

35. $y=2x-x^2,\ y=-x$.

36. $y^3=x,\ y=-2,\ x=8$.

37. $4x^2-9y+18=0$ and $2x^2-9y+36=0$.

38. $5x^2-60x+4y+160=0$ and $x^2-12x+2y+32=0$.

39. $x^2+y^2=8$ and $y^2=2x$.

40. $x^2+y^2=16$ and $y^2=4(x+1)$.

41. Draw the graph of the power function $y=x^m$ with m a positive integer; from an arbitrary point $A(x, y)$ of the curve thus obtained drop perpendiculars AB and AC on the coordinate axes. Prove that the area bounded by the curve $y=x^m$, the x-axis and the ordinate $x=x$ constitutes $\frac{1}{m+1}$th part of the area of the rectangle $OCAB$.

42. Prove that the area bounded by the continuous curve $x=$ $=\varphi(y)$, the y-axis and abscissas $y=a$, $y=b$ is equal to $\int\limits_a^b x\,dy=$

$$=\int\limits_a^b \varphi(y)\cdot dy.$$

43. Show that the area bounded by the arc AB of an equilateral hyperbola $xy=k$, by the x-axis and the ordinates of the points A and B is equal to the area bounded by this arc AB, the y-axis and the abscissas of the points A and B (each of the areas is equal to $k\cdot\ln\frac{b}{a}$).

44. Calculate the area of the segment of a parabola $x^2-12y=0$ cut off by a straight line passing through the coordinate origin and a point of the parabola with abscissa 6.

45. Calculate the area of the segment of a parabola $4y-x^2=0$ cut off by a chord joining points of the parabola with abscissas 2 and 4.

46. The equation of a curve is $y=x(3-x)^2$. Calculate the area bounded by this curve, the x-axis and the ordinates of the maximum and minimum points.

47. Find the volume of a paraboloid of revolution formed by revolving, about the x-axis, a segment (of altitude x) of the parabola $y^2=x$.

48. Find the volume of the spheroid obtained by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about its minor axis $(a>b)$.

49. Calculate the volume obtained by revolving one sinusoidal wave $y=\sin x$.

50. Calculate the volume obtained by revolving, about the x-axis, the area included between the parabolas $y^2=4x$ and $y^2=x^3$.

51. Calculate the volume obtained by revolving, about the x-axis, the area between the parabolas $y^2=8x$ and $y=x^2$.

52. A tangent is drawn to the parabola $y^2=12x$ at a point whose abscissa is equal to 6. Calculate the volume formed by revolving, about the x-axis, the area between the tangent, the x-axis and the parabola.

53. A tangent is drawn to the parabola $y^2=2(x-1)$ at a point whose abscissa is equal to 3. Find the volume obtained by revolving, about the x-axis, the area between the tangent, the x-axis and the parabola.

54. Part of the ellipse $\frac{x^2}{5}+y^2=1$ between perpendiculars drawn through the foci to the x-axis revolves about the x-axis. Determine the volume of the barrel thus obtained.

55. Prove that the volume of a ring obtained by the revolving of a circle about the axis Ox that does not cut it (Fig. 162) is equal to the product of the area of the circle which is the cross-section of the ring by the mean circumference of the ring.

Hint. Let the radius of the circle be r and the coordinates of its centre A be $(0, b)$. The equation of the circle will then be

$$x^2+(y-b)^2=r^2.$$

The equation yields two values of y:

$$y-b=\pm\sqrt{r^2-x^2},$$

$$y_1=b+\sqrt{r^2-x^2},\ y_2=b-\sqrt{r^2-x^2}.$$

y_1 are the ordinates of the points of the semicircle BCD, and y_2 are the ordinates of the points of the semicircle BED. The volume of the ring is calculated by formula (XXII).

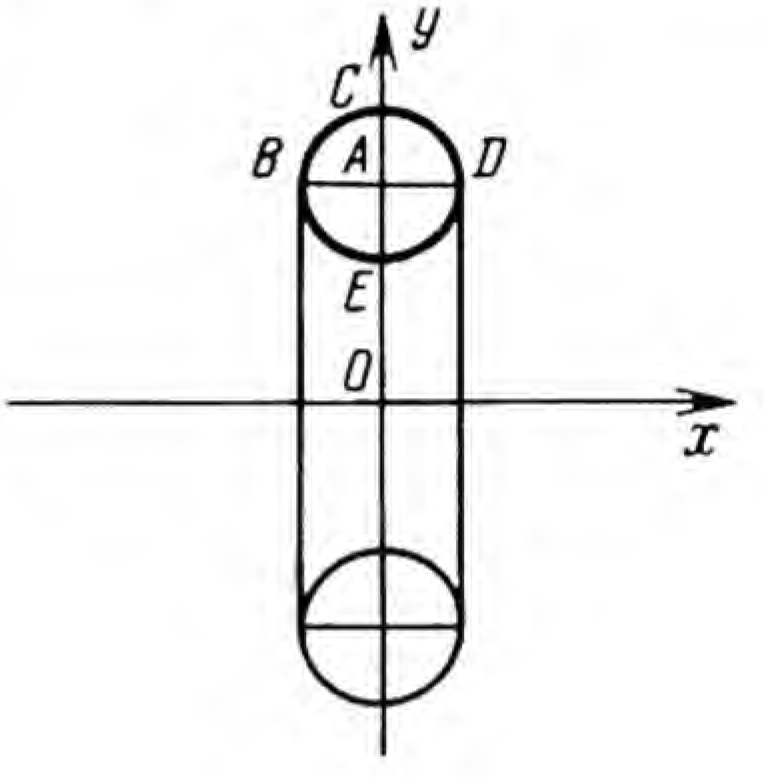

Fig. 162.

56. Find the volume of a body obtained by revolving a branch of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ about the x-axis from vertex $x=a$ to the section $x=x$, about the y-axis from section $y=0$ to section $y=y$.

57. Calculate the pressure experienced by a rectangular plate of canal locks 20 metres wide and submerged to a depth of 5 metres.

58. Calculate the pressure on the area of a triangle with base 8 metres and altitude 6 metres submerged in water so that the base lies on the surface of the water and the altitude is directed vertically downwards.

59. A vertical dam has the shape of a trapezoid, the top base of which is 50 metres, lower base 20 metres and altitude 10 metres. If the top base lies flush with the surface of the water, what is the pressure on the dam?

60. Determine the pressure on 1 dm^2 (decameter^2) of the vertical wall of the side of a ship if the centre of this square portion is 10 metres under water.

Hint (Fig. 163). The area of the rectangle $AB = ds = 1$ dm Δh dm (where dm=decameter) and it lies at a depth of $(100-h)$ dm from the surface of the water. Here, h is a positive number if AB lies above the centre O, and negative if AB lies below the centre O.

61. The end of a circular pipe lying horizontally in water may be closed by a damper. Determine the pressure on this damper if its diameter is 6 dm and its centre is 15 metres under water.

62. A body is in motion in a straight line in some medium according to the law $s = t^2$, where s is the distance travelled during time t. The resistance of the medium is proportional to the square of the velocity of motion. Determine the work done by the resistance of the medium as the body travels from $s = 0$ to $s = a$.

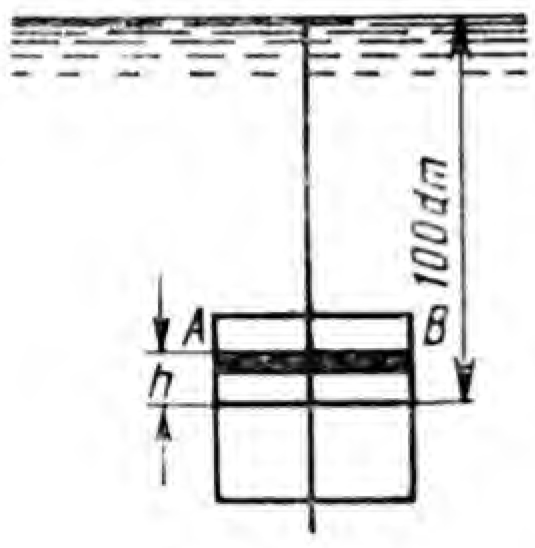

Fig. 163.

63. According to Hooke's law, the force necessary to stretch a metallic rod of length l_0 to length $l_0 + x$ is equal to $\frac{kx}{l_0}$, where k is a constant dependent on the properties of the metal. Calculate the work done in stretching the rod from length l_0 to l_1.

64. The compression of a spiral spring is proportional to the applied force. Calculate the work done in compressing the spring by 5 cm if a compression by 1 cm requires a force of 2 kg.

65. The force required to stretch a metallic spring is proportional to the tension. Calculate the work done in extending the spring by 5 cm if its extension by 1 cm requires a 10-kg force.

66. Calculate the work done in pumping out water from a cylindrical tank of radius r metres and depth h metres.

67. Calculate the work done in pumping out water from a full cauldron of semispherical shape with radius $r = 0.6$ metre.

68. According to Newton's law the force of gravitation is inversely proportional to the square of the distance. A body at rest attracts a material point moving in a straight line from the distance r_1 to r_2 measured from the centre of the attracting body. Calculate the work done by the force of gravitation.

Sec. 16. Differentiation of Functions of Several Variables

Find the partial derivatives, the partial differentials and the total differentials of the following functions:

1. $u = x \cdot y^2 \cdot z^3$.

2. $u = \frac{y}{x}$.

3. $u = (x + a)\ (y + b)$.

4. $u = \frac{x^2 + y^2}{x + y}$.

5. $u=(x^2+y^2)^n$.

6. $u=\sqrt{x^2+y^2+z^2}$.

7. $u=\ln(x^2+y^2)$.

8. $u=\ln(x^2+xy+y^2)$.

9. $u=\ln\frac{x+y}{x-y}$.

10. $u=\sin(x+y)$.

11. $u=\sin\frac{y}{x}$.

12. $u=\sin\frac{x+y}{z}$.

13. $u=\ln\sin\frac{x}{y}$.

14. $u=\ln\cos\frac{y}{x}$.

15. $u=e^{xy}$.

16. $u=e^{\frac{y}{x}}+e^{\frac{z}{x}}$.

17. $u=y^x$.

18. $u=x^{\sin y}$.

19. $u=\arctan(xy)$.

20. $u=\arcsin\frac{x}{y}$.

21. $u=\frac{xy}{x+z}$.

22. $u=xy\sin(x+y)$.

23. $u=(xy)^z$.

24. $u=z^{xy}$.

25. One leg of a right-angled triangle is increased from 6 cm to 6.2 cm and the other is diminished from 8 cm to 7.7 cm. What is the change in the hypotenuse (approximately)?

26. The volume of a truncated cone is expressed by the formula $v=\frac{1}{3}\pi h(R^2+r^2+Rr)$, where h is the altitude, R and r, the radii of the bases of the cone. Putting $R=30$ dm, $r=20$ dm and $h=40$ dm, find, approximately, the change in volume of the cone on an increase in R by 0.3 dm, in r by 0.4 dm and in h by 0.2 dm. State the percentage increase in volume.

27. Acceleration g is determined from the formula $s=\frac{1}{2}gt^2$. Determine the relative error in the calculated value of g if small errors were committed while measuring s and t.

28. The area of a triangle is given by the formula $s=\frac{1}{2}ab\cdot\sin C$. Determine the relative error in the calculated value of s if small errors were committed while measuring a, b and C.

Find the derivative $\frac{dy}{dx}$ of the following implicit functions:

29. $x^4+x^2y^2+y^4=0$.

30. $x^2+y^2=a(x^2-y^2)$.

31. $\sin x=\cos y$.

32. $\sin x=x\cdot\sin y$.

33. $y=1+x\cdot e^y$.

34. $x+\arctan y-y=0$.

35. $x=y-a\cdot\sin y$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

36. Find the equation of the tangent to the circle $x^2+y^2=a^2$ at the point (x_1, y_1).

37. Find the equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=$ $=1$ at the point (x_1, y_1).

38. Find the equation of the tangent to the parabola $y^2=2px$ at the point (x_1, y_1).

39. Tangents are drawn to the ellipse $\frac{x^2}{16}+\frac{y^2}{12}=1$ and to the hyperbola $x^2-\frac{y^2}{3}$ at their point of intersection (the hyperbola lies in the first quadrant).

Prove that the tangents are mutually perpendicular.

40. Find the equations of the tangents to the circle $y^2=2ax-x^2$ at points with abscissas a.

Sec. 17. Expansion of Functions in Power Series

Expand in powers of x:

1. $f(x)=a^x$, $(a>0)$.
2. $f(x)=\cos x$.
3. $f(x)=\ln(1-x)$.
4. $f(x)=\sin(a+x)$.
5. $f(x)=\sin^2 x$.
6. $f(x)=\cos^2 x$.
7. Expand e^x in powers of the difference $x-2$.
8. Expand $\ln x$ in powers of $x-1$.
9. Expand $f(x)=x^3-2x^2+5x-7$ in powers of $x-1$.
10. Expand $f(x)=(a+x)^m$ in powers of a.

Using known expansions, expand the following functions:

11. $f(x)=e^{-x^2}$.
12. $f(x)=\frac{1}{(1-x)^2}$.
13. Calculate $\sqrt[5]{1.2}$.
14. Expanding $\tan x$ in powers of $\left(x-\frac{\pi}{4}\right)$, calculate $\tan 46°$.

Sec. 18. Answers and Hints

Sec. 1

2. a) $(5,2)$; b) $(-5, -2)$. **3.** a) $(-a, -b)$; b) (a, b). **4.** a) 5 units of length; b) 3 units of length. **6.** Hint. $AC^2>AB^2+BC^2$. **7.** Two points are possible: $(-7, 4)$ and $(9, 4)$. **8.** $(0, 5)$ and $(0, -3)$. **9.** $(0, -3.4)$. **10.** $(4, 0)$. **11.** $(8,0)$ or $(-1, 3\sqrt{3})$. **12.** $(4, -3)$. **13.** $(1, -6)$. **14.** $\left(3, -1\frac{1}{2}\right)$, $\left(2\frac{1}{2}, -\frac{1}{2}\right)$, $\left(\frac{1}{2}, -4\right)$. **15.** $\left(0, 1\frac{2}{3}\right)$ and $\left(2, \frac{1}{3}\right)$ **16.** $(5.4, 2.8)$, $(7.8, 3.6)$, $(10.2, 4.4)$ and $(12.6, 5.2)$. **17.** a) $(-2, 0)$; b) $(0, 3)$; c) $(-2, -5)$;

d) (3, -1). **18.** (11, -5). **19.** (-12, 11). **20.** $\frac{3}{2}\sqrt{17}$. **21.** (-2, 1). Hint. The centre of gravity of a triangle is the point that divides the median in the ratio 2:1 measuring from the vertex. **22.** $\left(2\frac{2}{3},\ 5\frac{2}{3}\right)$. Hint. A bisector divides the opposite side in parts proportional to the adjoining sides. **23.** (3, 7), (-4, 0) and (1, -4). **24.** (2, -3). Hint. The diagonals of a parallelogram are bisected in their point of intersection; the required vertex is determined as the other end of a segment of which one end and the middle point are known. **25.** (1, -3) and (-4, 0).
26. $x=\frac{P_1x_1+P_2x_2+P_3x_3}{P_1+P_2+P_3}$, $y=\frac{P_1y_1+P_2y_2+P_3y_3}{P_1+P_2+P_3}$. Hint. First determine the coordinates of the point of application of the resultant of two forces, P_1 and P_2, and then the coordinates of the point of application of the resultant of $P_1+P_2+P_3$. It will be recalled that the resultant of two forces, P_1 and P_2, is equal to their sum and is applied at a point M lying on the segment M_1M_2 and dividing it into two parts that are inversely proportional to the forces. **27.** The formula does not give definite values for the coordinates of the point of application of the resultant since $P_1+P_2+P_3=6+(-18)+12=0$, i.e., the resultant is zero, and a zero resultant may be applied at any point in the plane. **28.** a) 45°; b) 135°; c) 0. **31.** In both cases, equal in absolute value and opposite in sign.

Sec. 2

1. a) $6x+8y+7=0$; b) $2x-5y=0$. **2.** a) $y=2$; b) $x=2\frac{1}{2}$. **3.** a) (3, 7); b) (5, 13). **4.** The first straight line passes through points A, C, E, the second, through points A, B, and D. Point A is the point of intersection of the given straight lines. The straight lines do not pass through the coordinate origin. **5.** a) $y=-\frac{5}{3}x+5$; b) $y=\frac{4}{7}x+4$. Hint. Find k by formula (VI), Sec. 5. **6.** a) $y=x+3$; b) $y=3-x$; c) $y=3$. **7.** a) $y=\frac{x}{\sqrt{3}}-5$; b) $y=-(x\sqrt{3}+5)$; c) $y=-5$. **8.** a) $y=\pm x$; b) 1) $y=-3$, $x=2$; 2) $y=-1$, $x=-5$; 3) $y=0$, $x=-3$; 4) $y=4$, $x=0$. **9.** $y=0.3x+2.7$. **10.** $y=-\frac{3}{4}x$. **11.** The straight line whose slope is equal to the velocity v, and the initial ordinate equal to s_0. **12.** 18 m/min. **13.** $y=\frac{P}{2}+\frac{p}{l}\cdot x$. Hint. The weight of the girder P is distributed equally on both supports, the weight of the man p is distributed on the supports in inverse proportion to his distance from each support. **14.** a) $k=-\frac{5}{12}$, $\varphi=157°23'$; b) $k=\frac{3}{4}$, $\varphi=36°52'$; c) $k=\frac{1}{2}$, $\varphi=26°31'$; d) $k=-\frac{3}{4}$, $\varphi=143°08'$. **15.** a) $k=1$, $b=2$; b) $k=-2$, $b=1$; c) $k=2$, $b=-\frac{1}{2}$; d) $k=-\frac{1}{2}$, $b=-\frac{1}{3}$; e) $k=\frac{1}{5}$, $b=0$; f) $k=0$, $b=-\frac{3}{2}$. **16.** a) 4 and -6; b) $\frac{2}{3}$ and 2. **19.** a) $\frac{x}{3}+\frac{y}{2}=1$;

b) $\frac{x}{-\frac{1}{2}}+\frac{y}{3}=1$; c) $\frac{x}{1}+\frac{y}{-1}=1$; d) $\frac{x}{-\frac{5}{3}}+\frac{y}{5}=1$. **20.** a) $\left(\frac{1}{3},\ 3\frac{2}{3}\right)$; b) $(-1,\ +1)$; c) the straight lines are parallel. **21.** $3x-4y\mp 12=0$ and $3x+4y\mp 12=0$. **22.** $x-y=0$ and $x+y+a=0$. **23.** $x+y-5=0$. **24.** $5x+3y-30=0$. **25.** a) $x-y-1=0$, b) $x+y-5=0$. **26.** $x\sqrt{3}\mp y\pm 2-6\sqrt{3}=0$. **27.** $(6,\ 0)$, $y=-\frac{2}{3}x+4$. **28.** a) $x+3y-5=0$; b) $2x-y+5=0$; c) $x=2$; d) $y=-3$. **29.** a) They do; b) they do not. **30.** a) 11; b) 3. **31.** a) 45°; b) 60°; c) 78°41′; d) 0; e) 90°. **32.** a) -2; b) $\sqrt{3}$; c) 0. **33.** a) 26°34′, 18°26′ and 135°; b) 45°, 90° and 45°. **34.** a) The first two are parallel to each other and perpendicular to the third; b) the first and third, parallel to each other and perpendicular to the second. **35.** a) $y=-2x$; b) $y=3x-15$; c) $y=-6$; d) $x=3$. **36.** a) $x+y=0$; b) $x-2y=0$; c) $y=0$; d) $x=0$. **37.** $y=2x$. **38.** $y=-2x$. **39.** $x+2y-10=0$ and $2x+4y+5=0$. **40.** $9x-y-16=0$ and $x+9y-20=0$. **41.** $x=3$ and $y=5$. **42.** $2x+y-7=0$; $x-2y-6=0$. **43.** $3x+4y-18=0$. **44.** $7x+35y-55=0$. **45.** a) $4x-y-6=0$; b) $x+4y+7=0$; c) $3x-5y-13=0$ and $5x+3y+1=0$. **46.** AB in the ratio $4:1$; CD in the ratio $2:3$. **47.** $4x-3y-17=0$. **48.** $(2,\ -3)$. **49.** a) 2; b) 3. **50.** a) 0; b) 5.2. **51.** (10, 21). **52.** 0.5. **53.** $3x-4y+5=0$ and $3x-4y-15=0$. Hint. The problem has two answers because the straight line lying 2 units of length from the given straight line can be drawn on either side of the given line. Hence, if the distance from one position of the straight line to the given line is $+2$, the other distance should be -2. **54.** $\frac{y-y_0}{x-x_0}=\frac{y_2-y_1}{x_2-x_1}$. **55.** $x-7y-2=0$. **56.** M (1, 0). Hint. The broken line AMB has minimum length if a ray from point A passes through M to point B in such a manner that the angle of incidence of the ray AM on the axis Ox is equal to its angle of reflection from that axis. **57.** 45°. **58.** $\frac{A_2x+B_2y+C_2}{A_2x_0+B_2y_0+C_2}-\frac{A_1x+B_1y+C_1}{A_1x_0+B_1y_0+C_1}=0$. Hint. Let x and y indicate the coordinates of the point of intersection, i.e., values which satisfy each of the given equations. Both equations represent one and the same straight line if their coefficients are proportional (Sec. 13, 2°), i.e., if $A_2x+B_2y+C_2=0$ and $\lambda(A_1x+B_1y+C_1)=0$. Hence, the equation of any line passing through the point of intersection of the given straight lines can be written in the form: $A_2x+B_2y+C_2-\lambda(A_1x+B_1y+C_1)=0$. λ is determined by the condition that a straight line passes through the point $M_0(x_0, y_0)$. **59.** $x+y-11=0$, and $3x-y-16=0$. **60.** a) $3x+y-7=0$, $3x+y-17=0$, $x-3y+1=0$ and $x-3y+11=0$; b) $7x-y-26=0$; $x+7y-18=0$; $7x-y+24=0$; $x+7y-68=0$. **61.** $(4,\ -6)$ or $(2.4,\ -6.8)$. **62.** The centre of the circle is $O(4, 5)$.

Sec. 3

1. a) $x^2+y^2-6x+10y+18=0$; b) $x^2+y^2+4x-2y=0$; c) $x^2+y^2+6x-32=0$. **2.** $x^2+y^2+3x-4y=0$. **3.** $3x^2+3y^2-16x-7=0$. **4.** $x^2+y^2-10y=0$. **5.** $x^2+y^2-4x-6y+4=0$. **6.** $x^2+y^2-2x+2y+1=0$; $x^2+y^2-10x+10y+25=0$. **7.** $(x+1)^2+y^2=16$ and $(x-3)^2+(y-4)^2=16$. **8.** $(x-1)^2+(y+6)^2=25$ and $(x-8)^2+(y-1)^2=25$. **9.** a) $x^2+y^2-8x+6y=0$; b) impossible, the points lie on a single straight line. **10.** $(x-4)^2+(y+3)^2=25$. **11.** a) $(0,\ 3)$, $r=3$; b) $(-4,\ 0)$, $r=5$; c) $(5,\ -2)$, $r=4$;

d) $\left(1\frac{1}{2}, 2\right)$, $r=\frac{3\sqrt{7}}{2}$; e) $\left(1\frac{1}{4}, -\frac{3}{4}\right)$, $r=\frac{3\sqrt{10}}{4}$; f) circle of zero radius with a real point $(6, -1)$; g) imaginary circle. **12.** $x^2+y^2+6x+5=0$. **13.** $x^2+y^2+3x-4y=0$. **14.** $x^2+y^2=49$. **15.** a) The circle touches the axis of abscissas at the point $(3, 0)$ and intersects the y-axis in points: $(0, 9)$ and $(0, 1)$; b) the circle touches the x-axis at the point $(5, 0)$ and does not intersect the y-axis. **16.** a) Intersect in points $(5, 0)$ and $(-3, -4)$; b) touch at point $(-3, -4)$; c) the straight line lies outside the circle. **17.** $a=\pm 30$; $b=48$.

Sec. 4

1. a) $\frac{x^2}{16}+\frac{y^2}{12}=1$; b) $\frac{x^2}{625}+\frac{y^2}{400}=1$; c) $\frac{x^2}{676}+\frac{y^2}{192}=1$. **2.** $\frac{x^2}{9}+y^2=1$. **3.** $\frac{x^2}{289}+\frac{y^2}{225}=1$. **4.** $\frac{4x^2}{289}+\frac{y^2}{16}=1$. **5.** $16x^2+25y^2=1600$. **6.** $\frac{x^2}{625}+\frac{y^2}{576}=1$. **7.** $\frac{x^2}{169}+\frac{y^2}{144}=1$. **8.** $9x^2+25y^2=225$. **9.** $9x^2+25y^2=225$. **10.** $8x^2+9y^2=162$. **11.** a) $x^2+4y^2=100$; b) $2x^2+3y^2=180$. **12.** a) $2a=26$, $2b=10$, $F(12, 0)$, $F_1(-12, 0)$, $e=\frac{12}{13}$; b) $2a=2\sqrt{10}$, $2b=2\sqrt{6}$, $F(2, 0)$, $F_1(-2, 0)$, $e=\frac{\sqrt{10}}{5}$; c) $2a=8$, $2b=8\sqrt{2}$, $F(0, 4)$, $F_1(0, -4)$, $e=\frac{\sqrt{2}}{2}$; d) $2a=\frac{4}{3}$, $2b=\frac{4}{5}$, $F\left(\frac{8}{15}, 0\right)$, $F_1\left(-\frac{8}{15}, 0\right)$, $e=0.8$; e) $2a=3$, $2b=5\frac{1}{3}$, $F\left(0, \frac{5}{6}\sqrt{7}\right)$, $F_1\left(0, -\frac{5}{6}\sqrt{7}\right)$, $e=\frac{5\sqrt{7}}{16}$. **13.** Ellipse $x^2+4y^2=36$. **14.** a) $\frac{\sqrt{2}}{2}$, b) $\frac{\sqrt{10}}{5}$; c) $\frac{1}{m}\sqrt{m^2-1}$. **15.** $\frac{b}{a}=\sqrt{1-e^2}$. **16.** $e\approx 0.08$. **17.** $2c=5.1\times 10^6$ km. **18.** $\left(-\frac{15}{2}, \frac{3\sqrt{7}}{2}\right)$ and $\left(-\frac{15}{2}, -\frac{3\sqrt{7}}{2}\right)$. **19.** Hint. Take the sides of the right angle as the coordinate axes, put segments AM and MB equal to a and b, then $\frac{x}{a}=\cos\varphi$ and $\frac{y}{b}=\sin\varphi$. To eliminate φ, square these equalities and add. **20.** a) $AB=5$, $AM=3$, $MB=2$; b) $AB=5$, $AM=4$, $MB=1$; c) $AB=10$, $AM=MB=5$. **21.** Hint. In triangle C_1OC_2 the hypotenuse $C_1C_2=r_2-r_1$ and the legs $OC_1=a-r_1$ and $OC_2=r_2-b$. By Pythagorean theorem we obtain the required relation in the form:
$(a-r_1)^2+(r_2-b)^2=(r_2-r_1)^2$. Removing brackets and adding $-2ab$ to both sides of the equation, we may represent the relation in the form:

$$(b-r_1)(r_2-a)=\frac{(a-b)^2}{2}.$$

22. Hint. The coordinates of the point M are

$$x=(a-b)\cdot\cos\varphi,\ y=(a+b)\cdot\sin\varphi.$$

Using the method indicated in problem 19, we get the equation

$$\frac{x^2}{(a-b)^2}+\frac{y^2}{(a+b)^2}=1.$$

Thus, the curve is an ellipse. A device like this made of two laths connected with a hinge screw is used for drawing big ellipses (e.g., for tracing arches).

Sec. 5

1. a) $\frac{x^2}{121}-\frac{y^2}{48}=1$; b) $\frac{x^2}{144}-\frac{y^2}{81}=1$; c) $\frac{4x^2}{49}-\frac{y^2}{8}=1$. 2. $x^2-\frac{y^2}{8}=1$. 3. $\frac{x^2}{25}-\frac{y^2}{75}=1$. 4. $\frac{x^2}{36}-\frac{y_2}{45}=1$. 5. $\frac{x^2}{25}-\frac{y^2}{144}=1$. 6. $9x^2-16y^2=144$. 7. $9x^2-8y^2=72$. 8. a) $F(\pm 10, 0)$, $e=\frac{5}{3}$, $y=\pm\frac{4}{3}x$; b) $F(\pm 4, 0)$, $e=0.4\sqrt{10}$, $y=\pm x\sqrt{0.6}$; c) $F\left(\pm 2\frac{5}{6}, 0\right)$, $e=\frac{17}{15}$, $y=\pm\frac{8}{15}x$. 9. a) $x^2-4y^2=8$; b) $9x^2-16y^2=20$. 10. $120°$. 11. $\frac{2}{\sqrt{3}}$, $\sqrt{2}$. 12. $e=\left|\frac{1}{\cos\varphi}\right|$. 13. a) 2 and 32; b) 34 and 4. 14. $x=9\frac{3}{5}$, $y=\pm\frac{3}{5}\sqrt{119}$ (2 points). 15. $x_0=\pm a\sqrt{2-\frac{1}{e^2}}$, $y_0=\pm b\sqrt{1-\frac{1}{e^2}}$ (4 points). Hint. Since the radii vectors joining the point (x_0, y_0) with the foci are perpendicular to each other, the sum of the squares of their lengths is equal to the square of the interfocal distance:

$$(ex_0-a)^2+(ex_0+a)^2=4c^2.$$

The abscissa x_0 is determined from this equation; substituting its values into the equation of the hyperbola, we find y_0. 16. $x^2-y^2=12$. 17. a) $xy=6$; b) $xy=4$. 18. a) $x^2-y^2=6$; b) $x^2-y^2=10$. 19. $9x^2-16y^2=144$. 20. $9x^2-7y^2=63$. 21. a) $a_4=\frac{2ab}{\sqrt{a^2+b^2}}$; b) $a_4=\frac{2ab}{\sqrt{b^2-a^2}}$ may be solved if $b>a$. 22. $4x\pm 3y-20=0$. 23. a) $\left(5, -5\frac{1}{3}\right)$. $\left(-3\frac{3}{4}, 3\right)$; b) at an infinitely remote point the straight line serves as an asymptote; c) does not intersect. 24. $(2, \pm 2)$ and $(-2, \pm 2)$. 25. $\left(5\frac{3}{5}, \pm\frac{6\sqrt{6}}{5}\right)$ and $\left(-5\frac{3}{5}, \pm\frac{6\sqrt{6}}{5}\right)$.

Sec. 6.

1. a) $y^2=16x$; b) $x^2=-12y$. 2. a) $y^2=-4x$; b) $x^2=8y$. 3. a) $y^2=4x$; b) $y^2=-8x$. 4. a) $2x^2-9y$; b) $3x^2=-4y$. 5. $|p|=2\frac{2}{3}$. 6. $p=\frac{10}{3}$. 7. 112 mm. 8. a) $\frac{x}{a}+\frac{y^2}{b^2}=1$; b) $\frac{x^2}{a^2}+\frac{y}{b}=1$. These are called equations

of a parabola in intercept form. 9. Hint. Taking the equation of the parabola in intercept form (see problem 8b), we have $\frac{4x^2}{l^2}+\frac{y}{f}=1$. Whence $y=$ $=f\left(1-4\frac{x^2}{l^2}\right)$. The width of each part of the span is $\delta=\frac{l}{2n}$. Abscissas and ordinates are measured from the middle (origin O), and $x_k=\delta_k=$ $=\frac{lk}{2n}$. The stanchion (vertical beam) has—for this abscissa—length $y_k=$ $=f\left(1-\frac{k^2}{n^2}\right)$, where $k=1, 2, \ldots, n-1$. The length of the diagonal beam d_k inclined towards the middle of the truss is determined by the formula for the distance between two points with coordinates $\left(\frac{lk}{2n}, 0\right)$ and $\left(\frac{l(k-1)}{2n}, y_{k-1}\right)$, where $y_{k-1}=f\left[1-\frac{(k-1)^2}{n^2}\right]$,

$$d_k=\sqrt{f^2\left[1-\frac{(k-1)^2}{n^2}\right]^2+\frac{l^2}{4\pi^2}}.$$

The length of the diagonal beam d'_k inclined towards the sides of the truss is determined similarly to the foregoing:

$$d'_k=\sqrt{f^2\left(1-\frac{k^2}{n^2}\right)^2+\frac{l^2}{4n^2}}.$$

Note that $d'_k=d_{k+1}$, i.e., two diagonal beams converging to a point of the parabola are equal. This property makes the parabolic trusses technically superior to trusses of other shapes.

Answer: $\delta=2.5$ m, $y_1=4.69$ m, $y_2=3.75$ m, $y_3=2.19$ m, $d_1=5.60$ m, $d_2=d'_1=5.31$ m, $d_3=d'_2=4.51$ m and $d_4=d'_3=3.32$ m. 10. Hint. The equation of the upper parabola: $y=(f+f')\left(1-4\frac{x^2}{l^2}\right)$; the equation of the lower parabola: $y'=f'\left(1-4\frac{x^2}{l^2}\right)$. For $x_k=\frac{lk}{2n}$ the length of the stanchion $z_k=y_k-y'_k=f\left(1-\frac{k^2}{n^2}\right)$. Lengths of the diagonals:

$$d_k=\sqrt{(y_{k-1}-y'_k)^2+\delta^2}=\sqrt{\left[f\left(1-\frac{(k-1)^2}{n^2}\right)+f'\frac{2k-1}{n^2}\right]^2+\frac{l^2}{4n^2}},$$

$$\delta'_k=\sqrt{(y_k-y'_{k-1})^2+\delta^2}=\sqrt{\left[f\left(1-\frac{k^2}{n^2}\right)+f'\frac{2k-1}{n^2}\right]^2+\frac{l^2}{4n^2}}.$$

Answer: $\delta=2.5$ m; $y_1=4.68$ m, $y_2=3.75$ m, $y_3=2.18$ m; $y'_1=2.81$ m, $y'_2-2.25$ m, $y'_3=1.31$ m, $z_1=1.87$ m, $z_2=1.50$ m, $z_3=0.87$ m; $d_1=3.32$ m, $d_2=3.48$ m, $d_3=3.49$ m, $d_4=3.32$ m; $d'_1=3.02$ m, $d'_2=2.67$ m, $d'_3=2.50$ m, d'_4 (chord) $=2.82$ m. 11. a) $(x-2)^2=8(y-3)$; b) $(x-3)^2=-12y$; c) $(y+2)^2=$ $=12(x-1)$; d) $y^2=-8(x-2)$. 12. a) $x^2-8x-12y+28=0$; b) x^2-6x+ $+8y+17=0$; c) $y^2+8x-2y-7=0$; d) $y^2+4x+4=0$. 13. a) $y^2-10x+25=$ $=0$; b) $x^2-4x-8y+12=0$; c) $y^2+6x-4y+7=0$. 14. a) $x^2-4x-4y=0$; b) $y^2+4x-6y+1=0$. 15. a) $O'(2, 1)$, $p=5$, axis $\parallel Ox$; b) $O'(0, -7)$, $p=3$, axis $\parallel Ox$; c) $O'(2, 0)$, $p=4$, the axis coincides with the negative direction of the axis Ox; d) $O'(3, 5)$, $p=-2$, axis $\parallel Oy$; e) $O'(4, -1)$,

$p=\frac{1}{2}$; axis $\parallel Oy$; f) $O'(-3, -9)$, $p=\frac{1}{2}$, axis $\parallel Oy$; g) $O'(1, 1)$, $p=-\frac{1}{2}$; axis $\parallel Oy$; h) $O'\left(\frac{1}{2}, \frac{1}{4}\right)$; $p=-\frac{1}{2}$, axis $\parallel Oy$. **16.** a) $(O'(-3, 0)$, $r=4$; b) $O'(5, -1)$, $r=5$. **17.** a) $O'(1, -2)$, $a=5$, $b=3$; b) $O'(-1, -1)$, $a=\sqrt{6}$, $b=\sqrt{5}$. **18.** a) $(16, \pm 8)$, b) $(8, \pm 8)$. **19.** $x=2$.

Sec. 7

2. 3. 3. $26°34'$. 4. $4x^2+4y^2-60x-60y+225=0$ and $64x^2+64y^2+240x-240y+225=0$. 6. $\frac{p}{2}$ and $p\sqrt{2}$; $y=\pm 2x\sqrt{2}$. 7. $25x^2+25y^2=114$. 8. $\frac{x^2}{49}+\frac{y^2}{36}=1$; $\frac{x^2}{9}-\frac{y^2}{4}=1$; $M_{1,2}\left(\frac{21}{\sqrt{13}}, \pm\frac{12}{\sqrt{13}}\right)$; $M_{3,4}\left(-\frac{21}{\sqrt{13}}, \pm\frac{12}{\sqrt{13}}\right)$. 9. $x+5y-3=0$.

Sec. 8

2. Hint. $\cos x=\sin\left(\frac{\pi}{2}-x\right)$. Putting $\frac{\pi}{2}-x=y$, we have: if $x\to\frac{\pi}{2}$, $y\to 0$. 3. Hint. $|\cos x-1|=|1-\cos x|=2\left(\sin\frac{x}{2}\right)^2<2\cdot\frac{x^2}{4}=\frac{1}{2}x^2$. 7. 2. 8. 1. 9. $-\frac{1}{3}$. 10. 0. 11. $\frac{4}{3}$. 12. $\frac{2}{5}$. 13. $\frac{1}{2}$. 14. $\frac{3}{2}a$. 15. -1. 16. -1. 17. $\frac{a}{k}$. 18. 0. 19. 0. 20. ∞. 21. -1. 22. $\frac{1}{2a}$. 23. $\frac{\sqrt{3}}{36}$ 24. 0.

Sec. 9

1. -3. 2. x^2+2 and $x^2+8x+14$. 3. $\frac{x^2-4x+5}{x^2-4x+4}$. 6. 2, -3. 7. $f(1)=0$, $f(-2)=0$. 8. $f(x)\cdot f(y)=f(x+y)$. 9. $f(x+y)=\frac{f(x)+f(y)}{1-f(x)\cdot f(y)}$. 11. a) $-2\leqslant x\leqslant +2$; b) $-1\leqslant x\leqslant 3$; c) $1\leqslant x\leqslant +\infty$; d) $-2<x<+2$. 13. $3x^2\Delta x+3x\times\Delta x^2+\Delta x^3=1.261$ m^3. 14. $3x^2\Delta x+3x\Delta x^2+\Delta x^3-2\Delta x=0.100601$. 15. $-\frac{2\Delta x}{(x-1+\Delta x)(x-1)}$ 16. $\log\left(1+\frac{\Delta x}{x}\right)$. 17. a) 0; b) 1; c) -1 and $+1$; d) -3. Hint. At $x=-7$ the fraction becomes $\frac{0}{0}$, it is necessary to divide through by $x+7$; e) $\frac{7}{3}$ f) 0. 19. Continuous. 20. Hint. $\tan x=\frac{\sin x}{\cos x}$.

Sec. 10

2. 12. 3. $b-2ct$; $t=\frac{b}{2c}$. 4. 1.00425 kcal. 5. $2\cdot 10^7$ erg. 6. $\frac{\pi d}{16}$. 7. a) 0; b) 6; c) -4; d) 4 and 2. 8. a) $3x^2$; b) 12; c) cannot. 9. a) $y=-\frac{1}{x_1^2}\cdot x+2y_1$; b) $y=-x+2$; c) $y=-4x-4$; d) cannot. 10. 0 and $\frac{2}{3}$. 11. $\arctan 3=71°34'$. 12. $90°$ and $\arctan\frac{3}{4}=36°52'$. 13. (2, 4).

1. $6(x-1)$. 2. $2x(10x^2+1)$. 3. x^3-x+1. 4. $\frac{3}{5}x^2-\frac{2}{3}x-l$. 5. $2x(2ax^2+b)$. 6. $3ax^2+2bx-c$. 7. $x^2(4-2x^2+x^4)$. 8. $-\frac{21x^5}{4}+5x^3-3x$. 9. $-\frac{4ax^3}{b}-\frac{bx}{a}$. 10. $\frac{3x^2}{a+b}-\frac{2x}{a-b}-\frac{1}{ab}$. 11. $ax^{a-1}-b\cdot x^{b-1}+3abx^{ab-1}$. 12. $2ax^{2a-1}-2x^{\log 2-1}\cdot\log 2$. 13. $kx^{k-1}-2\pi x^{\pi-1}$. 14. $21x^{\frac{5}{2}}+10x^{\frac{3}{2}}+3x^{\frac{1}{2}}$. 15. $-6x^{-3}+x^{-\frac{3}{2}}+1$. 16. $\frac{3}{2\sqrt{x}}-\frac{4}{3\sqrt[3]{x^2}}+6\sqrt[5]{x}$. 17. $2x\sqrt{3}+\frac{15\sqrt{x}}{2}$. 18. $-\frac{2}{x^3}+\frac{21}{x^4}$. 19. $\frac{4a}{3x^5}-\frac{6a^2}{x^3}$. 20. $\frac{1}{\sqrt{x}}+\frac{1}{4x\sqrt{x}}$. 21. $-2(1+35x)$. 22. $-4x(10x^3-3x+1)$. 23. $40x^3+33x^2+50x-1$. 24. $-2(1-x+3x^5)$. 25. $2acx+ad+bc$. 26. $3acx^2+2bcx+ad$. 27. $2(9x^2+x-1)$. 28. $2x(3x^4-28x^2+49)$. 29. $\frac{5(1-x^2)}{(1+x^2)^2}$. 30. $-\frac{2}{(1+x)^2}$. 31. $-\frac{6x^2}{(1+x^3)^2}$. 32. $\frac{ad-bc}{(cx+d)^2}$. 33. $\frac{2ax(k+l)}{(l-ax^2)^2}$. 34. $\frac{1}{a}-\frac{a}{x^2}+\frac{2x}{b^2}-\frac{2b^2}{x^3}$. 35. $\frac{2}{5}-\frac{5}{2x^2}$. 36. $\frac{1}{3}+\frac{12x}{(4-x^2)^2}$. 37. $3-\frac{27}{(2-x)^2}$. 38. $\frac{b^2}{(a-x)^2}-\frac{a^2}{x^2}$. 39. $-\frac{acnx^{n-1}}{(b+cx^n)^2}$. 40. $\frac{dz}{dt}=\frac{1-2t}{(t^2-t+1)^2}$. 41. $\frac{dz}{dt}=\frac{1-3t^2}{\sqrt{\pi}}$. 42. $F'(1)=16$. 43. $F'(a)=-\frac{1}{2a+(1+a)\sqrt{a}}$. 44. $F'(a)\ \frac{1+a}{2a\sqrt{a}}$. 45. $s'(0)=\frac{3}{25}$; $s'(2)=1\frac{2}{15}$. 46. $\varrho'(2)=-\frac{3}{25}$; $\varrho'(0)=1$. 47. $30x(3x^2+8)^4$. 48. $-84x^2(5-4x^3)^6$. 49. $3(2ax+b)\cdot(ax^2+bx+c)^2$. 50. $8x(2mx^2+n)\times(mx^4+nx^2+p)^3$. 51. $k(x^m+x^n)^{k-1}(mx^{m-1}+nx^{n-1})$. 52. $p(amx^{m-1}+bnx^{n-1})(ax^m+bx^n)^{p-1}$. 53. $\frac{x}{\sqrt{a^2+x^2}}$. 54. $\frac{2cx+b}{3\sqrt[3]{(a+bx+cx^2)^2}}$. 55. $4ab$. 56. $-\frac{7x}{\sqrt{1+x^2}}$. 57. $\sqrt{5}-\frac{3}{\sqrt{3x+5}}$. 58. $2(4x\sqrt{3}+15\sqrt{x})\times(x^2\sqrt{3}+5\sqrt{x^3})^3$. 59. $-\frac{42x}{(x^2-1)^4}$. 60. $\frac{40x}{(3-2x^2)^3}$. 61. $-\frac{ax}{(a^2+x^2)\sqrt{a^2+x^2}}$. 62. $-\frac{an}{\sqrt{x}(1+\sqrt{x})^{n+1}}$. 63. $\frac{x(3x^2-2)}{\sqrt{1-x^2}}$. 64. $\frac{2x(4x^2+5)}{3\sqrt[3]{(x^2+1)^2}}$. 65. $\frac{1}{2\sqrt{2x(2+\sqrt{2x})}}$. 66. $\frac{a}{4\sqrt{ax(a+\sqrt{ax})}}$. 67. $\frac{5bx^2+4x+ab}{2\sqrt{1+bx}}$. 68. $\frac{x(45x^2+16)}{\sqrt{1+5x^2}}$. 69. $\frac{2(4-x-x^2)}{(x-1)^5}$. 70. $-\frac{60x^2}{(5x-4)^4}$. 71. $(120x+161)\times(3x+5)^2(5x+4)^4$. 72. $x(36x+41)\cdot(3x-2)(2x+3)^2$. 73. $(x+a)^{m-1}\times(x+b)^{n-1}[(m+n)x+mb+na]$. 74. $\frac{(a+x)^{m-1}[(n-m)x+mb+na]}{(b-x)^{n+1}}$. 75. $\frac{1}{(1-x)\sqrt{x^2-1}}$. 76. $\frac{2x}{(x^2+1)\sqrt{x^4-1}}$. 77. $-\frac{a^2}{x^2\sqrt{a^2+x^2}}$. 78. $-\frac{3}{x^4\sqrt[5]{(1+x^5)^2}}$. 79. $\frac{x(4+x^3)}{2(1+x^3)\sqrt{1+x^3}}$. 80. $\frac{2(3-x^2)}{3\sqrt[3]{(1+x^2)^5}}$.

81. $\frac{nx^{n-1}(2+\sqrt{x})}{2(1+\sqrt{x})^{n+1}}$. 82. $-\frac{1}{2(x+\sqrt{x})\sqrt{1-x}}$. 83. k. 84. 1. 85. $\frac{5}{3}$
86. 1. 87. $n\cos nx$. 88. $nx^{n-1}x\cdot\cos x^n$. 89. $n\cdot\sin^{n-1}\cdot\cos x$. 90. $-ma^mx^{m-1}\sin(ax)^m$.
91. $-am\cos^{m-1}ax\cdot\sin ax$. 92. $-\frac{a}{\sin^2 ax}$. 93. $bn\tan^{n-1}bx\cdot\sec^2 bx$. 94. $4\cos^2 x$.
95. $\frac{dy}{d\varphi}=\tan^2\varphi$. 96. $\frac{dy}{d\varphi}=\frac{1}{2}\tan^2\frac{\varphi}{2}$. 97. $\frac{d\varrho}{d\theta}=-2a\sin 2\theta$. 98. $\frac{d\varrho}{d\theta}=$
$=-\frac{k\sin 2\theta}{\sqrt{\cos 2\theta}}$. 99. $\frac{dx}{dt}=r(1-\cos t)$. 100. $\frac{dy}{dt}=r\cdot\sin t$. 101. $\frac{dy}{dx}=$
$=6(\cos 3x-\sin 2x)$. 102. $\frac{dy}{dt}=A\cdot\omega\cdot\cos(\omega t+a)$. 103. $\frac{ds}{dt}=\frac{a}{t^2}\cdot\sin\frac{a}{t}$.
104. $\frac{ds}{dt}=-\frac{2}{t^3}\cdot\cos\frac{1}{t^2}$. 105 $\frac{dy}{dx}=-2\cdot\sin 4x$. 106. $\sin x$. 107. $-\frac{3a}{x^2}\times$
$\times\sin^2\frac{1}{x}\cdot\cos\frac{1}{x}$. 108. $-\sin x\cdot\cos(\cos x)$. 109. $-\cos x\cdot\sin(\sin x)$.
110. $-3\operatorname{cosec}^4 x$. 111. $\sec^6 x$. 112. $\tan^4\frac{x}{2}$. 113. $\sqrt{1+\sin 2x}+\sqrt{1-\sin 2x}$.
114. $\frac{\sin x-x\cdot\cos x}{\sin^2 x}$. 115. $\sqrt{2}\cdot\cos\left(\frac{\pi}{4}-2x\right)$. 116. $\sin x\cdot\cos 2x+\sin 3x$.
117. $2\sin 2x\cdot\cos^2 x$. 118. $f'\left(\frac{\pi}{3}\right)=-4\sqrt{3}$. 119. $f'\left(\frac{\pi}{6}\right)=4\sqrt{3}$.
120. $f'\left(\frac{\pi}{2}\right)=a$. 121. $f'(\pi)=\frac{2}{9}$. 122. $a\left(\tan ax+\frac{ax}{\cos^2 ax}\right)$. 123. $a\cdot\tan ax\times$
$\times\sec ax$. 124. $an\sec^n x\cdot\tan ax$. 125. a) $\ln 2=0.69315$, $\ln 7=1.94591$, $\ln 13=$
$=2.56495$; b) $\log 3=0.47712$, $\log 5=0.69897$, $\log 11=1.04139$. 126. $\frac{1}{(3+x)\ln a}$.
127. $\frac{2x}{(1+x^2)\ln a}$.
128. $\frac{2x}{x^2-a^2}$. 129. $\frac{3}{5+3x}$. 130. $\frac{2x}{3(a^2+x^2)}$. 131. $\frac{3}{x-5}+\frac{2}{x+4}$.
132. $\frac{3(1+\ln^2 x)}{x}$. 133. $\frac{1}{2x\cdot\ln a}$. 134. 0. 135. $\frac{1}{2x}+\frac{1}{2x\sqrt{\ln x}}$.
136. $\frac{1}{x\ln x}$. 137. $\frac{1}{2(x+1)}$. 138. $\frac{2a}{a^2-x^2}$. 139. $\frac{n}{x(1+x^n)}$.
140. $\frac{2(1+\log_a x)}{x\cdot\ln a}$. 141. $\frac{3\cdot\log_a^2 2x}{x\cdot\ln a}$. 142. $\frac{2x}{3(1+x^2)}$. 143. $\frac{1}{(1-x)\sqrt{x}}$.
144. $\frac{1+x}{x(x+\ln x)}$. 145. $\frac{an}{x}$. 146. $\frac{an\cdot\ln^{n-1}x}{x}$. 147. $2\cot 2x$.
148. $-\tan\frac{x}{2}$. 149. $2\cot x$. 150. $\cot 2x$. 151. $\frac{1}{\cos x}$. 152. $\frac{\cos\ln x}{x}$.
153. $\frac{1}{x\cdot\cos^2\ln x}$. 154. $2(x-1)\cdot 5^{x^2-2x}\cdot\ln 5=(x-1)\cdot 5^{x^2-2x}\cdot\ln 25$.
155. $(x^2-1)\cdot 2^{x^3-3x}\ln 8$. 156. $x\cdot 3^{1+x^2}\cdot\ln 3$. 157. $nx^{n-1}+n^x\cdot\ln n$.
158. $\frac{e^{2x}-1}{e^x}$. 159. $e^{\frac{x}{2}+\frac{2}{x}}\left(\frac{1}{2}-\frac{2}{x^2}\right)$. 160. $\frac{d\varrho}{d\varphi}=a^{\varphi}\cdot\ln a$. 161. $\frac{d\varrho}{d\varphi}=$
$=\frac{a^{\ln\varphi}\cdot\ln a}{\varphi}$. 162. $\frac{a\cdot e^{\sqrt{x}}}{2\sqrt{x}}$. 163. $a^{\frac{x}{e}}\cdot\ln a$. 164. $5^{\sin^2 x}\cdot\sin 2x\cdot\ln 5$.

165. $\frac{n\cdot a^{\tan nx}\cdot \ln a}{\cos^2 nx}$. 166. $-\frac{e^{\tan\frac{1}{x}}}{\left(x\cdot\cos\frac{1}{x}\right)^2}$. 167. $a^{e^x}\cdot e^x\cdot\ln a = a^{e^x}\cdot\ln a^{e^x}$.
168. $e^{a^x}\cdot a^x\cdot\ln a$. 169. $e^{2x}(2x-5)-4e^x(1+x)$. 170. $x^2\cdot e^x(\cos x-\sin x)+2x\cdot e^x\cdot\cos x$. 171. $\frac{1-\ln x}{x^2}$. 172. $\ln x$. 173. $e^{x\ln x}(1+\ln x)$
174. $\frac{4}{(e^x+e^{-x})^2}$. 175. $\frac{4\ln a}{(a^x+a^{-x})^2}$. 176. $-x\cdot\cot^2 x$. 177. $x\cdot\ln\tan x+\frac{x^2}{\sin 2x}+\cos x$. 178. $f'\left(\frac{\pi}{4}\right)=4\sqrt{2}$. 179. $f'\left(\frac{1}{2}\right)=\pi\cdot e^{\frac{\pi}{2}}$. 180. $2ae^{ax}\cdot\sin ax$.
181. $f'\left(\frac{\pi}{6}\right)=4$. 182. $f'(e)=\frac{1}{3e}$. 183. $f'(e)=\frac{4}{3}$. 184. $\frac{2}{x\cdot\ln x}$. 185. $f'(0)=1$.
186. $14x-1$. 187. $\frac{1}{\sqrt{1+x^2}}$. 188. $\frac{1}{2\sqrt{x(x+a)}}$. 189. $\frac{x+\sqrt{x^2-1}}{x\sqrt{x^2-1}}$.
190. $\frac{1}{\sqrt{a^2-x^2}}$. 191. $-\frac{1}{x^2+1}$. 192. $-\frac{1}{x\sqrt{x^2-1}}$. 193. $\frac{e^{\arcsin x}}{\sqrt{1-x^2}}$. 194. -1.
195. -1. 196. $\frac{1}{x(1+\ln^2 x)}$. 197. $\frac{1}{x(1+\ln^2 x)}$. 198. $\frac{1}{x}$. 199. $\frac{\cot x}{1+\ln^2\sin x}$.
200. $\frac{a}{a^2+x^2}$. 201. $\arcsin x+\frac{x}{\sqrt{1-x^2}}$. 202. x arc cot x. 203. $\sqrt{\frac{a-x}{a+x}}$.
204. $2\sqrt{1-x^2}$. 205. $\frac{2a^3}{x^4-a^4}$. 206. $x^{\frac{1}{x}}(1-\ln x)\frac{1}{x^2}$. 207. $(\sin x)^x(x\cdot\cot x+\ln\sin x)$. 208. $(\sin x)^{\sin x}\cdot\cos x\cdot(1+\ln\sin x)$. 209. $2\cdot x^{\ln x-1}\cdot\ln x$
210. 0. 211. $x^{\arcsin x-1}\left(\arcsin x+\frac{x\cdot\ln x}{\sqrt{1-x^2}}\right)$. 212. $e^{x^x}\cdot x^x\cdot(1+\ln x)$.
213. $n\left(\frac{x}{n}\right)^{nx}\cdot\left(1+\ln\frac{x}{n}\right)$.

Sec. 12

1. $-\infty<x<2$ decreases, $+2<x<+\infty$ increases. 2. $-\infty<x<1$ decreases, $+1<x<+\infty$ increases. 3. $-\infty<x<0$ and $+1<x<+\infty$ increases, $0<x<1$ decreases. 4. $-\infty<x<0$ and $+2<x<+\infty$ increases, $0<x<+2$ decreases. 5. $-\infty<x<-1$ and $0<x<1$ decreases, $-1<x<0$ and $+1<x<+\infty$ increases. 6. $-\infty<x<-\sqrt{2}$ and $0<x<\sqrt{2}$ decreases, $-\sqrt{2}<x<0$ and $+\sqrt{2}<x<+\infty$ increases. 7. $\frac{5}{4}$—minimum. 8. $\frac{3}{2}$—maximum. 9. $-\frac{3}{2}$—maximum. 10. -3—maximum, 3—minimum. 11. 0—minimum, $\frac{1}{3}$—maximum. 12. There is no extreme. 13. There is no extreme. 14. -2—maximum, $\frac{4}{3}$—minimum. 15. -1—maximum, $\frac{7}{3}$—minimum. 16. 0 and 2—minimum, 1—maximum. 17. -3 and 2—minimum, 1—maximum. 18. -1—minimum, 1—there is no extreme. 19. $\frac{1}{2}$—minimum, -1—there are no extremes. 20. $-\frac{1}{3}$ and 1—minimum, $\frac{1}{2}$—maximum, -1—there is no extreme. 21. -2.5—maximum, 2.5—minimum.

22. -1—maximum, 5—minimum. 23. -1—minimum, 1—maximum. 24. -1—maximum, 3—minimum. 25. -5—maximum, 1—minimum. 26. No extreme. 27. 3—minimum. 28. 2—minimum. 29. -4—maximum. 30. 0—maximum. 31. 0—minimum, 2—maximum. 32. e—minimum. 33. At $x=\frac{\pi}{4}$, maximum$=\sqrt{2}$ 34. At $x=\frac{\pi}{4}$, minimum$=2$; at $x=-\frac{\pi}{4}$, maximum-2. 35. No extreme. 36. At $x=\frac{\pi}{3}$, maximum $=\frac{3}{4}\sqrt{3}$. 37. At $x=\frac{\pi}{4}$, minimum $=\frac{\sqrt{2}}{4}$. 38. 5 and 5. 39. 9 and 1. 40. 1. 41. $\frac{1}{2}$. 42. Square. 43. Square with side $R\sqrt{2}$. 44. Equilateral with side $R\sqrt{3}$. 45. Isosceles. 46. Altitude of rectangle should be equal to radius of semicircle. 47. $R=\frac{1}{2}a$. 48. $2R=H$. 49. $R=\sqrt[6]{\frac{9v^2}{2\pi^2}}$, $H=R\sqrt{2}$. 50. $a\sqrt{2}$, $b\sqrt{2}$. 51. Breadth equal to $\frac{2}{3}$ altitude. 52. Three km from the village. 53. After 4 hours. 54. $\frac{a}{\sqrt{2}}=0.7a$. 56. Concavity upwards, $x=0$ minimum. 57. Concavity downwards at $x<0$ and upwards at $x>0$, cusp at $x=0$. 58. $x=0$ maximum, $x=1$ minimum, $x=\frac{1}{2}$ cusp, concavity downwards at $x<\frac{1}{2}$ and upwards at $x>\frac{1}{2}$. 59. See ex. 11; $x=\frac{1}{6}$ is a cusp; concavity upwards at $x<\frac{1}{6}$ and downwards at $x>\frac{1}{6}$. 60. See ex. 30; concave upwards in the intervals $-\infty<x<-\frac{\sqrt{2}}{2}$ and $\frac{\sqrt{2}}{2}<x<+\infty$ concave downwards in the interval $-\frac{\sqrt{2}}{2}<x<$ $<+\frac{\sqrt{2}}{2}$. 61. See ex. 21; concave upwards at $x>0$ and downwards at $x<0$, $x=0$ is a point of discontinuity. 62. See ex 25; concavity downwards at $x<-2$, upwards at $x>-2$, $x=-2$ is a point of discontinuity. 63. $x=0$ is a point of discontinuity, $x=1$ is a minimum point, $x=-\sqrt[3]{2}$ is a cusp, concavity upwards in the intervals $-\infty<x<-\sqrt[3]{2}$ and $0<x<+\infty$ and downwards in the interval $-\sqrt[3]{2}<x<0$. 64. Domain of definition $-\infty<x\leqslant 4$, maximum at $x=3$, concavity downwards. 65. $v_0=10$; $v_1=5$; $v_2=6$; $a_0=-8$; $a_1=-2$; $a_2=4$; $v_{\min}=4\frac{2}{3}$ at $t=\frac{4}{3}$. 66. $v_0=-ak$, $a_0=ak^2$. 68. $v=\frac{a\omega}{\cos^2\omega t}$, $a=-\frac{a\omega^2\tan\omega t}{\cos^2\omega t}$.

Sec. 13

2. $\Delta y=0.100601$; $dy=0.1$ 3. $\Delta y=-\frac{1}{2001}$; $dy=-\frac{1}{2000}$. 4. $(3ax^2+2bx+$ $+c)\,dx$. 5. $\left(\frac{1}{2\sqrt{x}}-\frac{1}{3x\sqrt[3]{x}}\right)dx$. 6. $\left(\frac{1}{c}+\frac{c}{x^2}\right)\cdot dx$. 7. $\frac{2x\,dx}{3\sqrt[3]{(1+x^2)^2}}$. 8. 0,0256. 9. -0.125. 10. $-\frac{2dx}{1-\sin 2x}=-\operatorname{cosec}^2\left(\frac{\pi}{4}-x\right)dx$. 11. $\tan^4 x\,dx$.

12. $\frac{5\,dx}{2x}$. 13. $(1+\ln x)\,dx$. 14. $\frac{x\,dx}{1+x^2}$. 15. $\frac{dx}{\sqrt{1-2x-x^2}}$. 16. $2e^x\cdot\cos x\,dx$. 17. 1 m^2, 4%. 18. 0.034 m^2; approximately 2.9%. 19. $\frac{1}{2}\left|\frac{dl}{l}\right|$.

Sec. 14

1. $\frac{1}{4}x^4+c$. 2. $\frac{3}{4}x\sqrt[3]{x}+c$. 3. $2\sqrt{x}+c$. 4. $-\frac{2}{\sqrt{x}}+c$. 5. $\frac{1}{5}x^5-\frac{3}{4}x^4+$ $+\frac{5}{3}x^3+c$. 6. $\frac{x}{2}(2+x-4x^2)+c$. 7. $\frac{1}{3}x^3+\frac{1}{x}+c$. 8. $\frac{2}{3}x\sqrt{x}+2\sqrt{x}+c$. 9. $2x+3\ln x+c$. 10. $\frac{1}{2}x^2+4x+4\ln x+c$. 11. $s=v_0t+\frac{1}{2}at^2$. 12. $y=\frac{3}{2}x^2-$ $-x+\frac{1}{2}$. 13. $\sqrt{2x}\left(\frac{2x}{3}-1\right)+c$. 14. $\ln\frac{1}{1-x}+c$. 15. $-\frac{1}{x+1}+c$. 16. $-\frac{1}{x^4+a^4}+c$. 17. $-\frac{1}{8(x^2+a^2)^4}+c$. 18. $\frac{a}{2}\ln(a^2+x^2)+c$. 19. $2\ln\times$ $\times(1+\sqrt{x})+c$. 20. Hint: take $\sqrt{x}$ out of the brackets. Answer: $\ln(\sqrt{x}-1)+c$.

21. $\frac{1}{2}(\ln x)^2+c$. 22. $2\sqrt{\ln x}+c$. 23. $\frac{1}{2}\ln\ln x^2+c=\frac{1}{2}\ln\ln x+c_1$, where $c=c_1+\frac{1}{2}\ln 2$. 24. $-\frac{e^{-4x}}{4}+c$. 25. $-\frac{1}{3\cdot e^x}+c$. 26. $6\cdot a^{2x}+c$. 27. $\frac{1}{8\ln 2}\cdot 2^{4x^2-8}+c$. 28. $\frac{x^3}{3}+4x^2+4x+\ln(x-1)+c$. 29. $\frac{x^3}{3}-x^2+$ $+3x+\ln(x+1)+c$. 30. $\frac{1}{3}(x^2-a^2)\sqrt{a^2-x^2}+c$. 31. $\frac{2}{9}(x^3-1)\times$ $\times\sqrt{x^3-1}+c$. 32. $\ln(x^2+ax+b)+c$. 33. $2\sqrt{x^2+ax+b}+c$. 34. $\ln(e^x+e^{-x})+c$. 35. $a\ln(e^{\frac{x}{a}}-e^{-\frac{x}{a}})+c$. 36. $2(1+e^x)\sqrt{1+e^x}+c$. 37. $\frac{1}{5}\sqrt{(2x-1)^5}+\frac{1}{3}\sqrt{(2x-1)^3}+c$. 38. Hint: $\int\frac{dx}{1+e^x}=\int\left(1+\right.$ $\left.+\frac{1}{1+e^x}-1\right)dx$. Answer: $x-\ln(1+e^x)+c$. 39. Hint. Rationalise the denominator of the fraction. Answer: $\frac{2}{3a}[\sqrt{(x+a)^3}+\sqrt{x^3}]+c$. 40. $\frac{2}{3}[\sqrt{(x+1)^3}-\sqrt{x^3}]+c$. 41. $\frac{x}{6}(3x+4\sqrt{x}+6)+c$. 42. $\frac{x}{20}(12\sqrt[3]{x^2}+$ $+15\sqrt[3]{x}+20)+c$. 43. $\ln\arcsin x+c$. 44. $\frac{1}{2}(\arctan x)^2+c$. 45. $\frac{2}{3}\times$ $\times\sqrt{\left(\frac{(x-1)}{x}\right)^3}+c$. 46. $\frac{1}{2}\ln\ln\frac{x^2}{1+x^2}+c$. 47. $s=t^3+t^2$. 48. $s=\frac{1}{12}t^4+2t$. 49. $\frac{a}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})$. 50. $-\frac{1}{n}\cos(nx+a)+c$. 51. $\frac{5}{2}\sin\frac{2x-3}{5}+c$. 52. $-\frac{1}{2}\cos 2x-\frac{1}{3}\sin 3x+c$. 53. $\frac{1}{a}\sin ax-a\cdot\cos\frac{x}{a}+c$. 54. $a\cdot\sin\frac{x}{a}+$ $+\frac{1}{a}\cos ax+c$. 55. $\tan x-\cot x+c$. 56. $\frac{1}{5}(\tan 5x+\cot 5x)+c$. 59. $\sec x+c$. 60. $-\operatorname{cosec} x+c$. 61. $\frac{2}{3}\sqrt{\tan^3 x}+c$. 62. $-2\sqrt{\cot x}+c$.

3. $\frac{1}{3}\sin^3 x+c$. 64. $-\frac{2}{3}\cos^3\frac{x}{2}+c$. 65. $-\frac{1}{\sin x}+c$. 66. $\frac{1}{2a\cdot\cos^2 ax}+c$.
67. $-\frac{1}{6}\cos(2x^3+3)+c$. 68. $-\frac{1}{3}\sin\frac{3}{x}+c$. 69. $\frac{1}{2}\tan x+c$. 70. $\frac{1}{2}\ln(1+2\sin x)+c$. 71. $\frac{1}{b}\ln(a-b\cos x)+c$. 72. $-\frac{1}{2(x+\sin x)^2}+c$.
73. $-4\cot\frac{x}{2}+c$. 74. $\frac{1}{a}\cdot\tan\frac{x}{2}+c$. 75. Hint. $\tan^2 x=\sec^2 x-1$. Answer: $-x+\tan x+c$. 76. $-x-\cot x+c$. 77. $-e^{\cos x}+c$.
78. $\frac{a^{\sin x}}{\ln a}+c$. 79. $\tan x+\sec x+c$ or $\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)+c$. 80. $-\frac{1}{2}\cot 2x+\frac{1}{2}\operatorname{cosec} 2x+c$. 81. $-\frac{1}{2}\sqrt{\cos 2x}+c$. 82. $-\sqrt{1+\cos^2 x}+c$.
83. $\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)+c$. 84. $2\sqrt{2}\cdot\sin\left(\frac{x}{2}-\frac{\pi}{4}\right)+c$. 85. $\ln\tan x+c$.
86. $-2\cdot\cot 2x+c$. 87. $\frac{1}{2}(1-\tan x)^2+c$. 88. $-\frac{1}{2}(1+\cot x)^2+c$.
89. $\ln(1+\sin x-2\cos x)+c$. 90. $\frac{1}{1-m}(x\cdot\sin x+\cos x-1)^{1-m}+c$.
91. $\sin x-\cos x$. 92. $\frac{1}{2}\sin 2t+6$. 93. $\frac{1}{6}\arctan\frac{2x}{3}+c$.
94. $\frac{1}{6}\arctan\frac{3x}{2}+c$. 95. $\frac{1}{\sqrt{35}}\arctan\sqrt{\frac{5}{7}}x+c$. 96. $\frac{1}{\sqrt{6}}\arctan\sqrt{\frac{3}{2}}x+c$.
97. $\frac{1}{12}\ln\frac{2+3x}{2-3x}+c$. 98. $\frac{1}{2\sqrt{35}}\ln\frac{\sqrt{7}+x\sqrt{5}}{\sqrt{7}-x\sqrt{5}}+c$. 99. $\arcsin\frac{x}{3}+c$.
100. $\arcsin\frac{x}{2}+c$. 101. $\frac{1}{\sqrt{7}}\arcsin\sqrt{\frac{7}{5}}x+c$. 102. $\frac{1}{\sqrt{5}}\arcsin\sqrt{\frac{5}{7}}x+c$.
103. $\ln(x+\sqrt{x^2\pm 9})+c$. 104. $\ln(x+\sqrt{x^2\pm 4})+c$.
105. $\frac{1}{2}\ln(2x+\sqrt{4x^2-3})+c$. 106. $\frac{1}{\sqrt{7}}\ln(x\sqrt{7}+\sqrt{1+7x^2})+c$.
107. $\frac{5}{2}\arcsin x^2+c$. 108. $\frac{1}{3}\arcsin\frac{x^3}{a}+c$. 109. $\arctan e^x+c$.
110. $\arcsin e^x+c$. 111. $\frac{1}{a}\arctan\frac{\sin x}{a}+c$. 112. $\frac{1}{2a}\ln\frac{a-\cos x}{a+\cos x}+c$.
113. $\arcsin x+\sqrt{1-x^2}+c$. 114. $a\cdot\arcsin\frac{x}{a}-\sqrt{a^2-x^2}+c$.
120. $\arcsin(\ln x)+c$. 121. $\frac{1}{\sqrt{ab}}\arctan\left(\sqrt{\frac{b}{a}}\tan x\right)+c$.
122. $\frac{1}{2\sqrt{ab}}\ln\frac{\sqrt{a}+\sqrt{b}\cdot\tan x}{\sqrt{a}-\sqrt{b}\cdot\tan x}+c$. 123. $\sin x-\frac{1}{3}\sin^3 x+c$.
124. $-\cos x+\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x+c$. 125. $-\frac{2}{3}\cos^3 x+\cos x+c$.

126. $\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + c$. **127.** $-\frac{2}{5}\cos^5 x + c$.

128. $\sin x + \frac{1}{2}\sin^2 x + c$, or $\frac{1}{2}(1+\sin x)^2 + c$, where $c = \frac{1}{2} + c_1$.

129. $-\cos x - \frac{1}{2}\sin^2 x + c$, or $\frac{1}{2}(1-\cos x)^2 + c_1$, where $c = \frac{1}{2} + c_1$.

130. $\frac{1}{2}x + \frac{1}{4a}\sin 2ax + c$. **131.** $\frac{1}{2}x - \frac{1}{4a}\sin 2ax + c$. **132.** $\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c$. **133.** $\frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + c$.

134. $\frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{16}\sin 4x + c$. **135.** $-\frac{1}{2}\cot^2 x - \ln\sin x + c$.

136. $\frac{1}{3}\tan^3 x - \tan x + x + c$.

Sec. 15

1. 45. **2.** 27. **3.** $\frac{1}{6}$. **4.** 15. **5.** $\frac{2}{3}(2\sqrt{2}-1)$. **6.** 2. **7.** $2\frac{2}{3}$. **8.** $\ln\sqrt{5}$. **9.** $e-1$. **10.** $\frac{a^2-1}{a\ln a}$. **11.** $\frac{\pi}{4}$. **12.** $\frac{\pi}{6}$. **13.** $\frac{1}{3}$. **14.** $\frac{\pi}{4}$. **15.** $\frac{\pi}{4}$. **16.** $\frac{4}{15}$. **17.** 16.5. **18.** $\frac{1}{2}gt^2 + 10t$. **19.** $x^4 - 16$. **20.** $\frac{1}{3}\ln\frac{1+x^3}{2}$. **21.** $-\ln\cos x$. **22.** $-\frac{\cos 2x}{2}$. **23.** 1. **24.** $\ln a$. Natural logarithm of numbers $a > 1$ indicates the area between the hyperbola $xy = 1$ and the x-axis within the limits from $x = 1$ to $x = a$; the logarithm is also called hyperbolic logarithm. **25.** 18. **26.** $12\frac{1}{4}$. **27.** 11. **28.** 4. **29.** $\frac{4}{3}$. **30.** $\frac{1}{6}$. **31.** 2. **32.** $\frac{32}{15}\sqrt{2}$. **33.** $2\frac{2}{3}$. **34.** $\frac{2}{3}$. **35.** 4.5. **36.** 32. **37.** 8. **38.** 8. **39.** $2\pi + \frac{4}{3}$. **40.** $\frac{16}{3}\pi + 4\sqrt{3}$. **44.** 3. **45.** $\frac{1}{3}$. **46.** 4. **47.** $\frac{1}{2}\pi y^2 x$, i.e., the volume of a paraboloid of revolution is equal to half the volume of a cylinder having the same base—a circle of radius y—and the same altitude x. **48.** $\frac{4}{3}\pi a^2 b$. **49.** π^2 cubic units. **50.** 4π. **51.** 9.6π. **52.** 72π. **53.** $\frac{4}{3}\pi$. **54.** $\frac{44}{15}\pi$. **55.** $\pi r^2 \cdot 2\pi b$. **56.** $\frac{\pi b^2}{3a^2}(x^3 - 3a^2 x + 2a^3)$; $\frac{\pi a^2}{3b^2}(y^3 + 3b^2 y)$. **57.** 250 tons. **58.** 48 tons. **59.** 1500 tons. **60.** 100 kg. **61.** 1350π kg $= 4241$ kg. **62.** $\frac{4}{3}ka^3$, where k is the coefficient of resistance. **63.** $\frac{k(l_1 - l_0)^2}{2e_0}$. **64.** 0.25 kgm. **65.** 1.25 kgm. **66.** $500\pi r^2 h^2$ kgm. **67.** Approximately 101.7 kgm. **68.** $\mu\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$, where μ is the constant of proportionality. Hint. The force has a minus sign since it is directed towards the centre of the body.

Sec. 16

1. $du = y^2z^3\,dx + 2xyz^3\,dy + 3xy^2z^2\,dz.$ 2. $du = \frac{x\,dy - y\,dx}{x^2}.$

3. $du = (y+b)\,dx + (x+a)\,dy.$ 4. $du = \frac{(x^2-y^2)(dx-dy) + 2xy\,(dx+dy)}{(x+y)^2}.$

5. $du = 2n\,(x^2+y^2)^{n-1}(x\,dx + y\,dy).$ 6. $du = \frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2+y^2+z^2}}.$

7. $du = \frac{2\,(x\,dx + y\,dy)}{x^2+y^2}.$ 8. $du = \frac{(2x+y)\,dx + (2y+x)\,dy}{x^2+xy+y^2}.$ 9. $du = \frac{2\,(x\,dy - y\,dx)}{x^2-y^2}.$ 10. $du = \cos(x+y)\,(dx+dy).$ 11. $du = \frac{x\,dy - y\,dx}{x^2} \times \cos\frac{y}{x}.$ 12. $du = \frac{z\,(dx+dy) - (x+y)\,dz}{z^2}\cos\frac{x+y}{z}.$ 13. $du = \frac{y\,dx - x\,dy}{y^2} \times \cot\frac{x}{y}.$ 14. $du = \frac{y\,dx - x\,dy}{x^2} \cdot \tan\frac{y}{x}.$ 15. $du = e^{xy}(x\,dy + y\,dx).$ 16. $du = \frac{1}{x}\left(e^{\frac{y}{x}}\,dy + e^{\frac{z}{x}}\,dz\right) - \frac{dx}{x^2}\left(y\cdot e^{\frac{y}{x}} + ze^{\frac{z}{x}}\right).$ 17. $du = y^{x-1} \times (y\ln y\,dx + x\,dy).$ 18. $du = x^{\sin y}\left(\frac{1}{x}\sin y\,dx + \cos y\ln x\,dy\right).$

19. $du = \frac{y\,dx + x\,dy}{1+x^2y^2}.$ 20. $du = \frac{y\,dx - x\,dy}{y\sqrt{y^2-x^2}}.$ 21. $du = \frac{yz\,dx + (x^2+xz)\,dy - xy\,dz}{(x+z)^2}.$ 22. $du = (x\,dy + y\,dx)\sin(x+y) + xy\cos(x+y)\,(dx+dy).$ 23. $du = yz\,(xy)^{z-1}\,dx + xz\,(xy)^{z-1}\,dy + (xy)^z\ln(xy)\,dz.$ 24. $du = z^{xy}\ln z\,(y\,dx + x\,dy) + xyz^{xy-1}\,dz.$

25. Decreases approximately by 0.12 cm. 26. $820\,\pi$, about 3.3%.

27. $\frac{dg}{g} = \frac{ds}{s} - 2\frac{dt}{t}.$ 28. $\frac{ds}{s} = \frac{da}{a} + \frac{db}{b} + \frac{dc}{\tan c}.$ 29. $\frac{dy}{dx} = -\frac{x\,(2x^2+y^2)}{y\,(x^2+2y^2)}.$

30. $\frac{dy}{y} = \frac{x}{y}\cdot\frac{a-1}{a+1}.$ 31. $\frac{dy}{dx} = 1.$ Hint. $\cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\cos^2 y}.$

32. $\frac{dy}{dx} = \frac{\cos x - \sin y}{x\cdot\cos y}.$ 33. $\frac{dy}{dx} = \frac{e^y}{2-y}.$ Hint. $xe^y = y-1.$

34. $\frac{d^2y}{dx^2} = -\frac{2}{y^3}\cdot\frac{dy}{dx} = -\frac{2\,(y^2+1)}{y^5}.$ 35. $\frac{d^2y}{dx^2} = -\frac{\alpha\cdot\sin y}{(1-\alpha\cdot\cos y)^3}.$

36. $xx_1 + yy_1 = a^2.$ 37. $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$ 38. $yy_1 = p\,(x+x_1).$ 40. $y = \pm a.$

Sec. 17

1. $a^x = 1 + \frac{x\ln a}{1} + \frac{x^2\ln^2 a}{1\cdot 2} + \frac{x^3\ln^3 a}{1\cdot 2\cdot 3} + \ldots \quad (-\infty < x < +\infty).$

2. $\cos x = 1 - \frac{x^2}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} - \frac{x^6}{1\cdot 2\ldots 6} + \ldots \quad (-\infty < x < +\infty).$

3. $\ln(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\ldots\ (-1\leqslant x<+1)$.

4. $\underset{(-\infty<x<+\infty)}{\sin(a+x)}=\sin a+\frac{x}{1}\cos a-\frac{x^2}{1\cdot 2}\sin a-\frac{x^3}{1\cdot 2\cdot 3}\cos a+\ldots$

5. $\sin^2 x=\frac{2x^2}{1\cdot 2}-\frac{2^3\cdot x^4}{1\cdot 2\cdot 3\cdot 4}+\frac{2^5\cdot x^6}{1\cdot 2\ldots 6}-\ldots\quad(-\infty<x<+\infty)$.

6. $\cos^2 x=1-\frac{2\cdot x^2}{1\cdot 2}+\frac{2^3\cdot x^4}{1\cdot 2\cdot 3\cdot 4}-\frac{2^5\cdot x^6}{1\cdot 2\ldots 6}+\ldots\ (-\infty<x<+\infty)$.

7. $e^x=e^2+e^2(x-2)+\frac{e^2}{1\cdot 2}(x-2)^2+\ldots$ 8. $\ln x=(x-1)-$ $-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3+\ldots(1\leqslant x\leqslant 2)$. 9. $-3+4(x-1)+$ $+(x-1)^2+(x-1)^3$. 10. $(a+x)^m=a^m+ma^{m-1}x+\frac{m(m-1)}{1\cdot 2}\times$ $\times a^{m-2}x^2+\frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}a^{m-3}x^3+\ldots$ 11. $e^{-x^2}=1-x^2+\frac{x^4}{1\cdot 2}-\ldots$ $(-\infty<x<+\infty)$. 12. $\frac{1}{(1-x)^2}=1+2x+3x^2+\ldots\quad(-1<x<1)$.

13. 1.037. 14. $\tan x=1+2\left(x-\frac{\pi}{4}\right)+\frac{4}{1\cdot 2}\left(x-\frac{\pi}{4}\right)^2+\frac{16}{1\cdot 2\cdot 3}\times$ $\times\left(x-\frac{\pi}{4}\right)^3+\ \ldots$ Assuming $x-\frac{\pi}{4}=46^\circ-45^\circ=1^\circ=\frac{\pi}{180}=0.017453$, we get $\tan 46^\circ=1.0355$.

F. A SHORT HISTORICAL NOTE

1°. The elements of higher mathematics studied in technical schools were first theoretically formulated in the seventeenth and eighteenth centuries. In 1637 Descartes published his geometry. Engels regarded this as a turning point in the development of mathematics.

"The turning point in mathematics was Descartes' *variable magnitude*. With that came *motion* and hence *Dialectics* in mathematics and *at once, too, of necessity the differential and integral calculus*, which moreover immediately begins, and which on the whole was completed by Newton and Leibniz, not discovered by them."*

However, the word "completed" should not be understood in the sense that in differential and integral calculus the last word had been said. Rather does it mean a completion of the work of the predecessors of Newton and Leibniz (Fermat, Descartes, Barrow and others who had employed the method of infinitesimals). Leibniz (1646-1716) brilliantly laid down the canons of this new calculus, thus completing the formulation of its basic principles in the form of a system of rules and formulas. Newton (1642-1727), by his discoveries in mechanics, confirmed the method of differential and integral calculus and pointed to a wide field of scientific and practical application.

2°. Mathematical analysis continued to develop in the following centuries and was enriched by new generalisations. The scholars of pre-revolutionary Russia and of the Soviet Union played a big role in this development. We shall here mention only those who have made inestimable contributions to world science.

An outstanding part in the development of analysis as well as other branches of mathematics was played by Academician

* Frederick Engels, *Dialectics of Nature*, Moscow 1954, pp. 341-42.

Leonard Euler (1707-1783) of St. Petersburg. "Read and reread Euler; he is the teacher of us all, "said Laplace (1749-1827), the great French mathematician, mechanician and astronomer. The scientific works of Euler embrace all branches of mathematics and are devoted to their systematisation and further development. They are characterised by simple and orderly exposition and by an originality and breadth of view. Euler worked out the techniques and methods of solving differential equations — this fundamental mathematical apparatus for the investigation of Nature's processes. To this day, they appear in all manuals dealing with mathematical analysis. Euler's works also embrace the subjects of mechanics, hydrodynamics, optics and astronomy.

The works of the St. Petersburg Academician Mikhail Vasilyevich Ostrogradsky (1801-1861) also occupy a prominent place in science. He made a number of important discoveries in integral calculus. All textbooks on analysis contain his method of integration of a certain type of rational fraction and his formula for the transformation of multiple curvilinear integrals.

The great geometer Nikolai Ivanovich Lobachevsky (1792-1856) also devoted a number of works to mathematical analysis, in which he expressed original views anticipating in many respects the scholars of the West.

Lobachevsky created a new geometry, the so-called non-Euclidean geometry. Lobachevsky's ideas made a deep impression on all initial principles and forms of structure of mathematics; they became guiding principles in all branches of the exact sciences—mechanics, physics, astronomy. They became very important (in some cases, of decisive importance) in questions of philosophy. The geometry created by Lobachevsky is to this day still fruitful; its multifarious applications are in the stage of development and intensive research.

A powerful mathematical school was founded by the brilliant mathematician Pafnuty Lvovich Chebyshev (1821-1894), Academician of St. Petersburg. He said: "There is an especially important problem amongst the vast number of problems: it is to apply the means at our disposal so as to extract the maximum possible advantage." It is precisely because of this that a "large portion of practical problems reduce themselves to finding maxima and minima of quantities. Such problems are entirely new to science, and only by solving them can we satisfy the demands of practice, which always seeks the best, the most profitable".

The need to solve practical problems (machine designing) led Chebyshev to the creation of a new branch of mathematics—the theory of the best approximation of functions. An outgrowth of this theory is the constructional theory of functions, which has very important applications. Chebyshev's activities covered a wide range. He not only solved many practical problems but also solved

theoretical problems of exceptional complexity. These include brilliant discoveries in the theory of numbers (the law of distribution of prime numbers) and in the theory of probability (law of large numbers). In integral calculus he gave a formula for the approximate calculation of definite integrals and showed that it is impossible to express certain integrals in terms of elementary functions.

Chebyshev's work in probability theory and the theory of approximation of functions was continued by the noted Soviet mathematician S. N. Bernshtein (b. 1880) and a number of his pupils.

In this century the scientific activities of Academician Nikolai Nikolayevich Luzin (1883-1950) led to the formation of the Moscow school of mathematics. It reached its greatest activity in Soviet times. In the scope and importance of the scientific problems handled and the results obtained, this school at present occupies first place in the world. Outstanding workers in the field of analysis are P. S. Alexandrov, A. N. Kolmogorov, D. Y. Menshov, I. I. Privalov, A. N. Tikhonov.

Soviet mathematics has grown immensely since the October Revolution of 1917. Not only the central cities (Moscow and Leningrad) but also other towns like Kiev, Tbilisi, Kharkov, Odessa, Saratov, Yerevan, etc., have become scientific centres. Here, many students have developed into outstanding mathematicians that are making new researches and discoveries.

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TO THE READER

Peace Publishers would be grateful for your comments on the content, translation and design of this book. We would also be pleased to receive any other suggestions you may wish to make. Our address is: 2, Pervy Rizhsky Pereulok, Moscow, U.S.S.R.

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